

Foundations of Reasoning

1 Logic

Syntax: A formula of propositional logic is constructed as follows:

- If p is an **atomic propositions**, then p is a formula.
- If P is a formula, then so is $\neg P$ (read “not P ”)
- If P and Q are formulas, then so is $P \wedge Q$ (read “ P and Q ”)
- If P and Q are formulas, then so is $P \vee Q$ (read “ P or Q ”)
- If P and Q are formulas, then so is $P \rightarrow Q$ (read “ P implies Q ” or “if P , then Q ”)

Let $Form$ be the set of all formulas.

Semantics: The **truth value** of an formula can be calculated from the following truth tables:

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Question. There are three suspects for a murder: Adams, Brown and Clark. Adams says “I didn’t do it. The victim was an old acquaintance of Brown’s. But Clark hated him.” Brown states “I didn’t do it. I didn’t even know the guy. Besides I was out of town all that week.” Clark says “I didn’t do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it.” Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it?

Answer. Consider three sentence symbols A , B and C with the following intended meanings:

A : “Adams speaks the truth”

B : “Brown speaks the truth”

C : “Clark speaks the truth”

We have the following information from the story:

1. There are two innocent men and they are telling the truth: $(A \wedge B) \vee (B \wedge C) \vee (A \wedge C)$
2. If Adams tells the truth, then Brown cannot be: $A \rightarrow \neg B$

3. If Brown tells the truth, both Adams and Clark must be lying: $B \rightarrow (\neg A \wedge \neg C)$
4. If Clark tells the truth, then Brown must be lying: $C \rightarrow \neg B$.

All 4 statements must be true simultaneously which leaves only one possibility (the gray row):

A	B	C	$(A \wedge B) \vee (B \wedge C) \vee (A \wedge C)$	$A \rightarrow \neg B$	$B \rightarrow (\neg A \wedge \neg C)$	$C \rightarrow \neg B$
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	F	T	T	T
F	T	T	T	T	F	F
T	F	F	F	T	T	T
T	F	T	T	T	T	T
T	T	F	T	F	F	T
T	T	T	T	F	F	F

Hence, Adams and Clark are truthful and Brown is not, so Brown is the killer.

Rules of Inference: A **rule of inference** is a function taking a sequence of formulas (the premises) and returning a formula (the conclusion), denoted $P_1, \dots, P_n \vdash Q$. An inference rule is **(classically) valid** if “any way of making the premises true also makes the conclusion true”. Explain why the following inference rules are valid:

- *Modus Ponens:* $P, P \rightarrow Q \vdash Q$
- *Disjunctive Syllogism:* $P \vee Q, \neg P \vdash Q$
- *Adjunction:* $P_1, P_2, \dots, P_n \vdash P_1 \wedge \dots \wedge P_n$
- *Monotonicity:* $P \rightarrow Q \vdash (P \wedge R) \rightarrow Q$;
 $P \vdash Q$ implies $P, R \vdash Q$

Given examples that demonstrate the failure of these rules.

2 Probability

Probability Measure. Suppose that W is a finite set of states. A probability measure assigns real values between 0 and 1 to subsets of W . That is, a **probability measure** is a function $p : \mathcal{E} \rightarrow [0, 1]$ where \mathcal{E} is the set of events (all subsets of W) that satisfies the Komogorov axioms.

Kolmogorov Axioms:

1. For each $E \in \mathcal{E}$, $0 \leq p(E) \leq 1$
2. $p(W) = 1$, $p(\emptyset) = 0$
3. If E_1, \dots, E_n, \dots are pairwise disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$), then $p(\bigcup_i E_i) = \sum_i p(E_i)$

Any probability measure p satisfies the following properties (*make sure you can explain why*)

- $p(\overline{E}) = 1 - p(E)$ (\overline{E} is the complement of E)
- If $E \subseteq F$ then $p(E) \leq p(F)$
- $p(E \cup F) = p(E) + p(F) - p(E \cap F)$

Conditional Probability: The probability of E given F , denoted $p(E|F)$, is defined to be

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

Bayes Theorem: $p(E|F) = p(F|E) \frac{p(E)}{p(F)}$

Bayes theorem is important because it expresses the quantity $p(E|F)$ (the probability of a hypothesis E given the evidence F) — which is something people often find hard to assess — in terms of quantities that can be drawn directly from experiential knowledge. For example, suppose you are in a casino and you hear a person at the next gambling table announce “Twelve”. We want to know whether he was rolling a pair of dice or a roulette wheel. That is, compare $p(\text{Dice} | \text{Twelve})$ with $p(\text{Roulette} | \text{Twelve})$. Based on our background knowledge of gambling we have $p(\text{Twelve} | \text{Dice}) = 1/36$ and $p(\text{Twelve} | \text{Roulette}) = 1/38$. Based on our observations about the casino, we can judge the prior probabilities $p(\text{Dice})$ and $p(\text{Roulette})$. But this is now enough to *calculate* the required probabilities.

Question: The Three Prisoners Paradox. Three prisoners A, B and C have been tried for murder and their verdicts will be told to them tomorrow morning. They know only that one of them will be declared guilty and will be executed while the others will be set free. The identity of the condemned prisoner is revealed to the very reliable prison guard, but not to the prisoners themselves. Prisoner A asks the guard “Please give this letter to one of my friends — to the one who is to be released. We both know that at least one of them will be released”.

An hour later, A asks the guard “Can you tell me which of my friends you gave the letter to? It should give me some clue regarding my own status because, regardless of my fate, each of my friends had an equal chance of receiving my letter.” The guard told him that B received his letter.

Prisoner A then concluded that the probability that he will be released is $1/2$ (since the only people without a verdict are A and C). But, A thinks to himself:

“Before I talked to the guard my chance of being executed was 1 in 3. Now that he told me B has been released, only C and I remain, so my chances of being executed have gone from 33.33% to 50%. What happened? I made certain not to ask for any information relevant to my own fate...”

Explain what is wrong with A ’s reasoning.

Answer. A reasoned as follows. Consider the following events:

G_A : “Prisoner A will be declared guilty” (we have $p(G_A) = 1/3$)

I_B : “Prisoner B will be declared innocent” (we have $p(I_B) = 2/3$)

We have $p(I_B \mid G_A) = 1$: “If A is declared guilty then B will be declared innocent.” Using Bayes Theorem,

$$p(G_A \mid I_B) = p(I_B \mid G_A) \frac{p(G_A)}{p(I_B)} = 1 \cdot \frac{1/3}{2/3} = 1/2$$

But, A did not receive the information that B will be declared innocent, but rather that “the guard said that B will be declared innocent.” So, A should have conditioned on the event:

I'_B : “The guard said that B will be declared innocent”

Given that $p(I'_B \mid G_A)$ is $1/2$ (given that A is guilty, there is a 50-50 chance that the guard could have given the letter to B or C). This gives us the following correct calculation:

$$p(G_A \mid I'_B) = p(I'_B \mid G_A) \frac{p(G_A)}{p(I'_B)} = 1/2 \cdot \frac{1/3}{1/2} = 1/3$$

Question: Here is a deductively valid argument:

Argument 1: If X is true, then Z is true. If Y is true, then Z is true. Therefore, if $X \vee Y$ is true, Z is true.

Explain why this argument is valid and give an example to illustrate why humans may not necessarily reason according to this inference rule. Here is an argument about conditional probabilities that looks analogous:

Argument 2: Conditional on X , the probability of Z is q . Conditional on Y , the probability of Z is q . Therefore, conditional on $X \vee Y$, the probability of Z is q . More formally, the argument is: If $Pr(Z \mid X) = q$ and $Pr(Z \mid Y) = q$, then $Pr(Z \mid X \vee Y) = q$.

1. Give an example (set of states with a probability measure) that provides a counter-example to Argument 2. A complete probability model specifies a probability for each possible state-description.
2. Give a “real-world” example of a probabilistic situation (sampling, etc.) that provides a counter-example to Argument 2.