# 3 The relationship between preferences and choice

The assumption that there exists a firm relationship between the preferences of the economic unit and its behavior (choice from the set of the feasible alternatives it faces) is very common in economic theory. In fact, the exclusive working supposition in the analysis of the economic unit's behavior is that it can be assigned a system of preferences, such that its choice is the best alternative according to that system of preferences. It should be noted that under this assumption, which is often called the **rationality principle**, the system of preferences assigned to the economic unit is conceived as a device (theoretical construction) that enables explanation and prediction of its behavior. The existence of such a tight relationship between preferences and choice is not necessary. The system of preferences need not be well defined, and still a meaningful choice may exist. It is also possible that the system of preferences is well defined, but is inconsistent with the observed choice of the economic unit. And it is also possible that the notion of preferences is questionable or even meaningless as, for example, when it is assigned to a group of individuals who face the task of making a collective choice. This chapter is devoted to a formal presentation of the concepts of **preference** and **choice** and to a clarification of the relationship between them. In particular, we wish to answer the following two questions:

1. What are the necessary properties of a system of preferences that ensure the existence of choice consistent with that system of preferences? The challenge here is the identification of the properties of preferences that ensure the existence of rational choice.

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This chapter draws on and adapts material from Chapter 1 in Austen-Smith and Banks (1998).

2. What are the necessary properties of choice that ensure that it can be conceived as consistent with some system of preferences? The challenge here is the identification of the properties of choice that ensure its rationalizability.

### 3.1 PREFERENCE-DRIVEN CHOICE

In the context of preference and choice, the basic concepts or primitives are the set of alternatives $X$ and the preference relation $R$, which is defined on it. In this chapter we assume that $X$ is a finite set and its elements are denoted by $x$, $y$, $z$, etc. The binary preference relation $R$, the relation "preferred or indifferent to," enables comparison between pairs (not necessarily all pairs) of alternatives by the economic unit; an individual, or a group of individuals. We write $xRy$ when alternative $x$ is preferred or indifferent to alternative $y$ from the perspective of the individual or the group. The strict preference relation $P$ and the indifference relation $I$ are defined using $R$ as follows:

$$xPy \leftrightarrow xRy \, \& \sim yRx$$

$$xIy \leftrightarrow xRy \, \& \, yRx$$

$\sim yRx$ means that $yRx$ is not satisfied. $xPy$ and $xIy$ mean, respectively, that "$x$ is preferred to $y$" and "$x$ is indifferent to $y$."

Let $\chi$ denote the set of all non-empty subsets of $X$.

$$\chi = \{S \subseteq X : S \neq \phi\}$$

Given the preference relation $R$ and the subset of alternatives $S$ that belongs to $\chi$, let us define the **maximal set** $M$ associated with $(R, S)$ as follows:

$$M(R, S) = \{x \varepsilon S : \forall y \varepsilon S, \, xRy\}$$

$M(R, S)$ includes the best or top-ranked alternatives in $S$ according to the preference relation $R$. Usually, $M(R, S)$ is called the **choice set** corresponding to $R$ and $S$, because it is natural to assume that, given $S$ and $R$, the chosen alternatives are those included in $M(R, S)$. That is, the decision maker selects from the feasible alternatives that he faces

the best alternative from his point of view. This assumption, which is common in economics and to some extent in other social sciences, is called the **rationality principle**. The behavioral pattern that it implies is called **rational behavior**.

The first question that we examine is: What are the necessary properties of a preference relation $R$ that ensure, for any given subset of alternatives $S$, that the choice set $M(R, S)$ is non-empty?

## Reflexivity

A preference relation $R$ is reflexive if every alternative $x$ is preferred or indifferent to itself, that is, $xRx$.

## Completeness

A preference relation $R$ is complete if for every two distinct alternatives $x$ and $y$, $x$ is preferred or indifferent to $y$, or $y$ is preferred or indifferent to $x$, that is, $xRy$ or $yRx$.

Verify that reflexivity and completeness of $R$ are necessary properties for ensuring that the choice set $M(R, S)$ is non-empty for every subset $S$.

## Transitivity

A preference relation $R$ is transitive if for every three alternatives $x$, $y$ and $z$ in $X$,

$$xRy \ \& \ yRz \rightarrow xRz$$

## Acyclicity

A preference relation $R$ is acyclic if for every series of alternatives in $X$,

$$\{x, \ y, \ z \ldots u, \ v\},$$

$$xPy \ \& \ yPz \ldots \& \ uPv \rightarrow xRv$$

Notice that acyclicity is a weaker requirement than transitivity; transitivity implies acyclicity, but acyclicity does not imply transitivity.

A relation $R$ is called an **ordering** if it is reflexive, complete and transitive. If $R$ is an ordering, then $M(R, S)$ is non-empty, for any given subset of alternatives $S$ that belongs to $\chi$. In other words, reflexivity, completeness, and transitivity are sufficient conditions for the existence of a non-empty choice set under all circumstances (any given $S$). This claim follows from the following result, which clarifies what are the necessary and sufficient conditions for the existence of a non-empty choice set for any $S$ in $\chi$.

**Theorem 3.1:** Let $R$ be reflexive and complete. Then the choice set $M(R, S)$ is not empty given any subset of alternatives $S$ in $\chi$, if and only if $R$ is acyclic.

**Proof:**[1] (sufficiency) For any $S \subset X$, choose $x \in S$. If for all $s \in S$, $xRs$, then the proof is complete; otherwise, since $R$ is complete and reflexive, there must exist an alternative $y$, $y \in S \setminus \{x\}$, such that $yPx$. If for all $s \in S$, $yRs$, then again the proof is complete. Otherwise there must exist $z$, $z \in S \setminus \{x, y\}$, such that $zPy$. Since $R$ is acyclic, in such a case it follows that $zPx$. Since $X$ (and therefore $S$) is a finite set, the same argument can be reapplied to conclude that there must exist an alternative that is preferred or indifferent to any other alternative in $S$.

(necessity) Suppose that $x_1 P x_2 P x_3 \ldots P x_{n-1} P x_n$. We have to prove that $x_1 R x_n$. Let $S = \{x_1, \ldots, x_n\}$ and suppose that $M(R, S) \neq \phi$. Since $x_{i-1} P x_i$, $i = 2, \ldots, n$, we get that $x_i \notin M(R, S)$, $i = 2, \ldots, n$. Therefore, since $M(R, S) \neq \phi$, it must be that $x_1 \in M(R, S)$, which implies that alternative $x_1$ is preferred or indifferent to any alternative in $S$, and in particular, $x_1 R x_n$, as we wish to prove. **Q.E.D.**

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[1] This chapter is part of the introduction to the book. To avoid excessive cumbersomeness and digression from the main topic of the book, we chose to include in this chapter only the central results that deal with the relationship between preference and choice. With the exception of the proof of Theorem 3.1, we do not present proofs of the results. A more comprehensive discussion that includes the proofs of the results presented in this chapter as well as additional results appears in Chapter 1 of Austen-Smith and Banks (1998).

A function $C(.)$ that specifies for any set of alternatives $S$ in $\chi$ a non-empty subset of alternatives in $S$, $C(S) \subset S$, is called a **choice function**. The chosen subset can include one or several alternatives. If the chosen subset always includes a single alternative, the choice function is called **resolute**. By Theorem 3.1, if $R$ is a reflexive, complete and transitive (and therefore acyclic) relation on $X$, that is, if $R$ is an ordering on the set of alternatives $X$, then $C(S) = M(R, S)$ is a well-defined choice function. This function is referred to as the **rational choice function corresponding to $R$, or the choice function driven by $R$.**

## 3.2 RATIONALIZABLE CHOICE

Let us turn to the properties of choice, observable properties that can be verified empirically, that ensure the rationalizability of choice by a reflexive, complete, and transitive relation. Namely, we consider the possibility of relating to actual choice as driven by some preference ordering R. For any subset of alternatives $S$, this ordering would generate the chosen alternatives as its maximal set $M(R, S)$.

A choice function is **rationalizable** if there exists a relation $R$ on $X$ such that for any $S$ in $\chi$, $C(S) = M(R, S)$. A natural candidate to provide rationalization to $C(.)$ is the relation $R_c$, which is derived from the choice function $C(.)$ as follows:

$$\forall x, y \in X, \, x R_c y \Leftrightarrow x \in C(\{x, y\})$$

The relation $R_c$ is called the **base relation** of $C(.)$. It can be easily verified that

**Proposition 3.1:** A choice function $C(.)$ is rationalizable if and only if it is rationalizable by its base relation. That is, if and only if for any $S$ in $\chi$, $C(S) = M(R_c, S)$.

The following example clarifies that not every choice function is rationalizable.

**Example 3.1**

Suppose that $X = \{x, y, z\}$, $C(\{x\}) = \{x\}$, $C(\{y\}) = \{y\}$, $C(\{z\}) = \{z\}$, $C(\{x, y\}) = C(\{x, z\}) = \{x\}$ and $C(\{y, z\}) = C(\{x, y, z\}) = \{y\}$.

By Proposition 3.1, since $C(\{x, y\}) = \{x\}$ and $C(\{x, y, z\}) = \{y\}$, every relation $R$ that rationalizes the choice function $C(.)$ and, in particular, the base relation $R_c$ must satisfy the impossible requirement: $xPy$ and $yPx$. This means that $C(.)$ is not rationalizable. ∎

Two properties of a choice function are presented below. The first property $\alpha$ is called consistency in contraction. The other property $\beta$ is called consistency in expansion. Consistency in contraction (expansion) is stated in terms of permissible changes in choice following contraction (expansion) in the set of alternatives.

A choice function satisfies **property** $\alpha$ if and only if for any $S$ and $T$ in $\chi$,

$$S \subseteq T \Rightarrow C(T) \cap S \subseteq C(S)$$

Property $\alpha$ implies that if alternative $x$ is chosen from $T$ and $T$ is contracted to $S$ by eliminating some alternatives that differ from $x$, then $x$ must be among the chosen elements from the contracted set of alternatives $S$. It can be easily verified that the choice function in Example 3.1 does not satisfy property $\alpha$. The choice function in this example is not exceptional. Some commonly used choice functions violate this property. For example, the plurality rule $C^{PL}(.)$ is inconsistent in contraction. According to this choice function, when a group of individuals selects a candidate from a set of candidates, every individual indicates who is his most preferred candidate and the chosen candidates are those who are most preferred by the largest number of individuals (of course, with a large number of voters the chosen candidate is usually unique). The following example clarifies that indeed the plurality rule violates property $\alpha$.

**Example 3.2**

Suppose that $T = \{x, y, z\}$ and the transitive preference relations of five voters are as follows: Two individuals prefer $x$ to $y$ and $y$ to $z$. Two individuals prefer $y$ to $z$ and $z$ to $x$. The fifth individual prefers $z$

to $x$ and $x$ to $y$. In this case, candidates $x$ and $y$ are chosen from $T = \{x, y, z\}$; however, only candidate $x$ is chosen from the contracted set $S = \{x, y\}$. The fact that candidate $y$ is not chosen from the contracted set $S = \{x, y\}$ means that the plurality rule violates property $\alpha$. ∎

A choice function satisfies **property $\beta$** if, and only if, for any $S$ and $T$ in $\chi$,

$$S \subseteq T \ \& \ C(S) \cap C(T) \neq \phi \Rightarrow C(S) \subseteq C(T)$$

Property $\beta$ implies that if alternative $x$ is chosen from $S$ and from $T$ that contains $S$, then any other alternative that is chosen from $S$ is also chosen from $T$. It can be easily proved that a resolute choice function satisfies property $\beta$ and that the plurality rule does not satisfy property $\beta$.

Properties $\alpha$ and $\beta$ are necessary and sufficient conditions for the rationalizability of a choice function $C(.)$ by an ordering.

**Theorem 3.2:** A choice function $C(.)$ is rationalizable by an ordering if, and only if, it satisfies properties $\alpha$ and $\beta$.

By Proposition 3.2 and the fact that a resolute choice function satisfies property $\beta$, it follows that property $\alpha$ is a necessary and sufficient condition for the rationalizability of a resolute choice function by an ordering.

The consistency properties $\alpha$ and $\beta$ are equivalent to the weak axiom of revealed preference (WARP) that is usually encountered in the context of consumer theory. A choice function satisfies the **weak axiom of revealed preference (WARP)** if, and only if, for any $S$ and $T$ in $\chi$,

$$x \in C(S), \ y \in S \setminus C(S) \ \& \ y \in C(T) \Rightarrow x \notin T$$

That is, if alternative $x$ is revealed preferred to alternative $y$ (if $x$ is chosen from $S$ and $y$ is contained in $S$, but not chosen), then alternative $y$ cannot be revealed preferred to alternative $x$ (the fact that $y$ is chosen from $T$ implies that $x$ does not belong to $T$).

**Proposition 3.2:** A choice function $C(.)$ satisfies properties $\alpha$ and $\beta$ if, and only if, it satisfies the weak axiom of revealed preference (WARP).

The concluding result of this chapter is obtained from Proposition 3.2 and Theorem 3.2.

**Theorem 3.3:** A choice function $C(.)$ is rationalizable by an ordering if, and only if, it satisfies the weak axiom of revealed preference (WARP).

### 3.3 EXERCISES

Chapter 1 in Austen-Smith and Banks (1998) presents a comprehensive discussion of the relationship between preference and choice. In this chapter we have presented the basic concepts and five important results relevant to this discussion.

### 3.1 Preference-driven choice

**Question 3.1**
1. What is the choice set in the theory of the consumer?
2. What is the choice set in the theory of the producer?

**Answers**
1. In consumer theory the choice set includes affordable bundles of commodities that are preferred or indifferent to any bundle of commodities in the consumer's budget set.
2. In producer theory the choice set consists of feasible production plans (activities) that yield maximal profit.

**Question 3.2**
Explain why reflexivity and completeness of $R$ are necessary properties for the choice set $M(R, S)$ to be non-empty for any $S$.

**Answer**
Reflexivity is a necessary property for ensuring the existence of a non-empty choice set in those cases where $S$ consists of a single alternative. Completeness is a necessary property for ensuring the

existence of a non-empty choice set in those cases where $S$ includes two alternatives.

## Question 3.3

Explain why reflexivity, completeness, and transitivity ensure the realization of the rationality principle.

## Answer

These three properties ensure that in any subset of alternatives there exists a "best" (most preferred) alternative. This means that the rationality principle can be realized.

## 3.2 Rationalizable choice

## Question 3.4

Prove Proposition 3.1, namely that a choice function $C(.)$ is rationalizable if and only if it is rationalizable by its base relation. That is, if and only if for any $S$ in $\chi$, $C(S) = M(R_c, S)$.

## Answer

The sufficiency part is simple; if the base relation of $C(.)$ rationalizes $C(.)$, then by definition, $C(.)$ is rationalizable. To prove the necessity part, suppose that $C(.)$ is rationalizable by some relation $R$ and let us prove that $R_c$ must be such an $R$. For any two alternatives $x$ and $y$ in X, by definition of $M(R,.)$, $xRy \Leftrightarrow x \in M(R, \{x, y\})$. Since $R$ rationalizes $C(.)$,

$$x \in M(R, \{x, y\}) \Leftrightarrow x \in C(\{x, y\}).$$

But, by definition of $R_c$,

$$x \in C(\{x, y\} \Leftrightarrow xR_cy$$

Hence, for any two alternatives $x$ and $y$ in X, $xRy \Leftrightarrow xR_cy$, that is, $R = R_c$.

### Question 3.5

Prove that a resolute choice function is consistent in expansion, that is, it satisfies the property $\beta$.

### Answer

If the choice function $C(.)$ is resolute, then the fact that the intersection of the choice sets $C(S)$ and $C(T)$ is non-empty implies that $C(S) = C(T)$. This means that $C(.)$ satisfies property $\beta$.

### Question 3.6

A group of voters selects a candidate from a group of candidates by applying the plurality rule. According to this rule, every voter indicates who is his most preferred candidate and the chosen candidate is the one who is most preferred by the largest number of voters. Discuss the following claim: "It is clear that the plurality rule satisfies property $\alpha$. If a certain candidate is chosen from a group of candidates $T$, then he is also chosen from any sub-group of candidates S which is contained in $T$, because in both cases the same candidate is the most preferred one for the same number of voters."

### Answer

The claim is false. It is true that if a certain candidate is chosen from the group $T$, then he is the most preferred candidate in $T$ for the largest number of voters. But when $T$ is reduced to the sub-group $S$, it is perfectly possible that the same candidate is no longer the one who is most preferred by the largest number of voters, as illustrated in Example 3.2.

### Question 3.7

Prove that the plurality rule is inconsistent in expansion, that is, it violates property $\beta$.

### Answer

Suppose that $T = \{x, y, z, w\}$ and the transitive preference relations of five voters are as follows. Two voters prefer $x$ to $y$, $y$ to $w$ and $w$ to $z$. Two voters prefer $w$ to $y$, $y$ to $z$ and $z$ to $x$. The fifth voter prefers

$z$ to $x$, $x$ to $w$ and $w$ to $y$. In this case candidates $x$ and $y$ are chosen from $S = \{x, y, z\}$, but x and w are chosen from $T = \{x, y, z, w\}$. The fact that candidate $y$ is not chosen from $T$ implies that the plurality rule violates property $\beta$.

## Question 3.8

Prove that the plurality rule does not satisfy the weak axiom of revealed preference (WARP).

### Answer

In Example 3.2, $x \in C(S)$, $y \in S \setminus C(S)$ & $y \in C(T)$, but $x \in T$, that is, the plurality rule violates WARP.

## Question 3.9

Suppose that $C(S)$ is a choice function driven by the preference relation $R$, $C(S) = M(R,S)$. Prove that $C(.)$ satisfies the consistency properties $\alpha$ and $\beta$.

### Answer

Suppose that $S \subseteq T$. By definition, $C(T) = \{x \in T : \forall y \in T, xRy\}$. Hence, if $z \in C(T) \cap S$, then $\forall y \in S, zRy$, that is, $z \in C(S)$. We have therefore obtained that the choice function $C(.)$ satisfies the consistency property $\alpha$,

$$S \subseteq T \Rightarrow C(T) \cap S \subseteq C(S).$$

Now suppose that $S \subseteq T$ and that $C(S) \cap C(T) \neq \phi$. Hence, if $z \in C(S)$, then $\forall y \in T, zRy$, that is, $z \in C(T)$. We have therefore obtained that the choice function $C(.)$ satisfies the consistency property $\beta$,

$$S \subseteq T \& C(S) \cap C(T) \neq \phi \Rightarrow C(S) \subseteq C(T)$$

## Question 3.10

1. Is a choice function which is driven by a preference relation R necessarily resolute?
2. Is the plurality rule a resolute choice function?

### Answer

1.  No. It is certainly possible that the maximal set corresponding to a particular set of alternatives contains more than a single alternative. Such examples are frequently encountered in the theory of the consumer.

2.  No. See example 3.2.

### 3.4 SUMMARY

*   In the context of preference and choice, the basic concepts or primitives are the **set of alternatives** $X$ and the **preference relation** $R$, which is defined on it.
*   Given the preference relation $R$ and the subset of alternatives $S$, the **maximal set** $M(R, S)$ includes the best alternatives in $S$ according to $R$.
*   The assumption that given $S$ and $R$, the chosen alternatives are those included in $M(R, S)$ is called the **rationality principle**. This assumption is common in economics and therefore $M(R, S)$ is also called the **choice set** corresponding to $R$ and $S$.
*   **Theorem 3.1:** Let $R$ be reflexive and complete. Then the choice set $M(R, S)$ is not empty given any subset of alternatives $S$ in $\chi$, if and only if $R$ is acyclic.
*   A relation $R$ is called an **ordering** if it is reflexive, complete and transitive.
*   A **choice function** $C(.)$ specifies for any set of alternatives $S$ a non-empty subset of alternatives in $S$, $C(S) \subset S$.
*   A choice function is **rationalizable** if there exists a relation $R$ on $X$ such that for any $S$ in $\chi$, $C(S) = M(R, S)$.
*   The **base relation** $R_c$ of a choice function $C(.)$ is defined as follows:
    $\forall x, y \in X, x R_c y \Leftrightarrow x \in C(\{x, y\})$.
*   **Proposition 3.1:** A choice function $C(.)$ is rationalizable if, and only if, it is rationalizable by its base relation.
*   **Property $\alpha$** (consistency in contraction): A choice function satisfies **property $\alpha$** if, and only if, for any $S$ and $T$ in $\chi$,

$$S \subseteq T \Rightarrow C(T) \cap S \subseteq C(S)$$

- **Property $\beta$** (consistency in expansion): A choice function satisfies **property $\beta$** if, and only if, for any $S$ and $T$ in $\chi$,

$$S \subseteq T \,\&\, C(S) \cap C(T) \neq \phi \Rightarrow C(S) \subseteq C(T)$$

- **Theorem 3.2:** A choice function $C(.)$ is rationalizable by an ordering if, and only if, it satisfies properties $\alpha$ and $\beta$.
- A choice function satisfies the **weak axiom of revealed preference (WARP)** if, and only if, for any $S$ and $T$ in $\chi$,

$$x \in C(S), \, y \in S \backslash C(S) \,\&\, y \in C(T) \Rightarrow x \notin T$$

- **Proposition 3.2:** A choice function $C(.)$ satisfies properties $\alpha$ and $\beta$ if, and only if, it satisfies the weak axiom of revealed preference (WARP).
- **Theorem 3.3:** A choice function $C(.)$ is rationalizable by an ordering if, and only if, it satisfies the weak axiom of revealed preference (WARP).