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Another characterization of the majority rule

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Abstract

Given any (finite) society confronting two alternatives, May [Econometrica 20 (1995) 680] characterizes the majority rule in terms of anonymity, neutrality and positive responsiveness. This final condition is usually criticized to be too strong. Thus, we drop it and give a similar characterization in terms of anonymity, neutrality, Pareto optimality and a condition we call weak path independence.

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1. Introduction

Given a society confronting two alternatives only, what is an appropriate aggregation rule, which will derive a social preference at every profile of individual preferences? May (1952) characterizes the majority rule in terms of three axioms, namely, anonymity, neutrality and positive responsiveness. Positive responsiveness is criticized for being 'too strong'.[1] In particular, take any two alternatives x and y and any preference profile from which the derived social preference is an indifference. Positive responsiveness requires that even when a single individual who was originally indifferent between x and y now favours x to y (while the others' preferences are unchanged), the new social preference must strictly favour x.

An attempt to drop the positive responsiveness condition is due to Maskin (1995) who characterizes the majority rule in terms of anonymity, neutrality and some 'maximal transitivity' condition. This

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[1]To quote from Campbell and Kelly (2000), "this is a very strong condition, and it is not at all clear why it should be imposed".

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characterization is in a setting where there may exist more than two alternatives and an aggregation rule defined over pairs of alternatives is applied to every pair so as to derive a social preference. It is known, since the famous voting paradox first pointed by de Condorcet (1785) that this may lead to an intransitive, even cyclic social preference. However, Maskin (1995) shows that, majority does the best among all anonymous and neutral aggregation rules defined over pairs of alternatives in the following sense: if majority does not generate a transitive social ordering at some profile, then no such rule may generate it. Nevertheless, given any such rule other than majority, there are preference profiles where this rule does not lead to a transitive social ordering but majority does, hence the maximal transitivity condition. The result is an odd number of agents with indifferences in individual preferences being ruled out.[2] Campbell and Kelly (2000) follow the way paved by Maskin (1995) and extend his theorem to any number of agents using another set of axioms.

We give, as in May (1952), a direct characterization of the majority rule, without referring to axioms defined on the social ordering generation process with more than two alternatives. Hence, we characterize the majority rule on a set of two alternatives, by preserving anonymity and neutrality, dropping positive responsiveness and using two additional axioms, namely Pareto optimality and weak path independence.

Section 2 gives the preliminaries and Section 3 states the characterization theorem and a corollary.

2. Preliminaries

Let $\boldsymbol{A}=\{a, b\}$ be a set of alternatives. For each positive integer n, we define a society $N=\{1, \dots, n\}$ and assume that every $i \in N$ has a complete and transitive preference $R_i \in \{-1, 0, 1\}$ over $\boldsymbol{A}$.[3] We denote by $R=(R_1, \dots, R_n) \in \{-1, 0, 1\}^n$ an n-tuple of these binary relations reflecting a preference profile of the society. A social welfare function is a function F: $\cup_{n\in\mathbb{N}}\{-1, 0, 1\}^n \to \{-1, 0, 1\}$ which gives a complete and transitive ordering of $\boldsymbol{A}$ for every preference profile of any society.

Now, take any two positive integers n, $n' > 0$ and consider any two disjoint societies $N=\{1, \dots, n\}$ and $N'=\{n+1, \dots, n+n'\}$. Given any two preference profiles $R \in \{-1, 0, 1\}^n$ and $R' \in \{-1, 0, 1\}^{n'}$, we define an operation $\oplus$ as follows:

$$R \oplus R' = (R_1, \dots R_n, R_{n+1}, \dots R_{n+n'}) \in \{-1, 0, 1\}^{n+n'}.$$

Thus, if R is the preference profile of the society N and R' is the preference profile of the society N', then $R \oplus R'$ is the preference profile of the 'joint' society $N \cup N'$.

Finally, we recall three standard conditions one can impose on social welfare functions:

Neutrality (N). Given any integer $n > 0$ and any $R \in \{-1, 0, 1\}^n$, we have $F(-R) = -F(R)$.

Anonymity (A). Given any integer $n > 0$, any $R \in \{-1, 0, 1\}^n$ and any permutation function Π: $N \to N$, we have $F(R_1, \dots R_n) = F(R_{\Pi(1)}, \dots R_{\Pi(n)})$.

[2] Dasgupta and Maskin (1998) give a similar result for a continuum of agents.

[3] We write $R_i = 1$ (resp. $R_i = -1$) whenever agent i strictly prefers a to b (resp. b to a). $R_i = 0$ means that i is indifferent between a and b.

Pareto Optimality (PO). Given any integer $n>0$ and any $R=(R_1,\ldots R_n)\in\{-1,0,1\}^n$ with $R_i\geq 0$ (resp. $R_i\leq 0$) for all $i\in N$ and $R_j=1$ (resp. $R_j=-1$) for some $j\in N$, we have $F(R)=1$ (resp. $F(R)=-1$).

3. The majority rule

A social welfare function F is said to be the majority rule if and only if given any integer $n>0$ and any $R\in\{-1,0,1\}^n$ we have $F(R)=\operatorname{sgn}(\Sigma_{i\in N} R_i)$.[4]

May (1952) characterizes the majority rule in terms of anonymity, neutrality and an additional axiom called positive responsiveness (PR) defined for every integer $n>0$ as follows:

For any $R, R'\in\{-1,0,1\}^n$ with $R'_i\geq R_i$ for all $i\in N$ and $R'_j>R_j$ for some $j\in N$, we have $F(R)\in\{0,1\}\Rightarrow F(R')=1$. Similarly, for any $R, R'\in\{-1,0,1\}^n$ with $R'_i\leq R_i$ for all $i\in N$ and $R'_j<R_j$ for some $j\in N$, we have $F(R)\in\{-1,0\}\Rightarrow F(R')=-1$.

Hence we know that a social welfare function F satisfies A, N and PR if and only if it is the majority rule. We give another characterization of the majority rule by dropping PR and using two additional axioms. One of these is PO and the other one is some 'path independence' condition. We define path independence as follows:

Take any two positive integers $n, n'>0$ and consider any two disjoint societies $N=\{1,\ldots,n\}$ and $N'=\{n+1,\ldots,n+n'\}$. A social welfare function F is said to be path independent (PI) if and only if for any $R\in\{-1,0,1\}^n$ and any $R'\in\{-1,0,1\}^{n'}$, we have $F(R\oplus R')=F(F(R)\oplus F(R'))$.[5]

So, take any social welfare function satisfying PI and consider any two disjoint societies. It does not matter whether you aggregate the individual preferences of each society separately, thus obtaining a 'representative' from each society and then aggregate the preferences of these representatives or you directly aggregate the individual preferences of the 'joint' society. However, PI is too strong to be satisfied by a social welfare function[6] and we will use its weaker version, which we define as follows:

Take any two positive integers $n, n'>0$ and consider any two disjoint societies $N=\{1,\ldots,n\}$ and $N'=\{n+1,\ldots,n+n'\}$. A social welfare function F is said to be weakly path independent (WPI) if and only if for any $R\in\{-1,0,1\}^n$ and any $R'\in\{-1,0,1\}^{n'}$ with $|F(R)-F(R')|\neq 2$, we have $F(R\oplus R')=F(F(R)\oplus F(R'))$.

WPI is by definition weaker than PI as it imposes the same requirement only when the 'representatives' of the disjoint societies are not in a total disagreement, i.e., one does not have the inverse of the other's preference.

We add PO and WPI on top of A and N to reap the following characterization theorem:

Theorem 3.1. *A social welfare function* $F:\cup_{n\in\mathbb{N}}\{-1,0,1\}^n\rightarrow\{-1,0,1\}$ *satisfies A, N, PO and WPI if and only if it is the majority rule.*

Proof. The 'if' part which states that the majority rule satisfies A, N, PO and WPI is obvious and left

[4] Given any real number $r\in\Re$, $\operatorname{sgn}(r)$ equals 1, 0, -1 when $r>0$, $r=0$, $r<0$, respectively.

[5] We owe the name of our condition to its analogy to the path independence condition motivated by Arrow (1963), introduced by Plott (1973) and further elaborated by Sertel and van der Bellen (1979) in the choice theory context.

[6] We will formally state this in our Corollary 3.1.

to the reader. To show the 'only if' part, take any social welfare function F which satisfies A, N, PO and WPI. We will show that F is the majority rule. Take any integer $n > 0$ and consider the society $N = \{1, \ldots, n\}$. For any $R \in \{-1, 0, 1\}^n$, write $n_+(R) = \#\{i \in N | R_i = 1\}$ for the number of people who strictly prefer a to b at the preference profile R. Similarly $n_-(R) = \#\{i \in N | R_i = -1\}$. To show that F is the majority rule, we must show that for any $R \in \{-1, 0, 1\}^n$, we have

(i) $n_+(R) = n_-(R) \Rightarrow F(R) = 0$
(ii) $n_+(R) > n_-(R) \Rightarrow F(R) = 1$
(iii) $n_+(R) < n_-(R) \Rightarrow F(R) = -1$

To show (i), take any $R \in \{-1, 0, 1\}^n$ with $n_+(R) = n_-(R)$. $F(R) = 0$ follows from the fact that F is anonymous and neutral. To show (ii), take any $R \in \{-1, 0, 1\}^n$ with $n_+(R) > n_-(R)$. Let $k = n_+(R) - n_-(R)$. Now take some coalition $K \subset \{i \in N | R_i = 1\}$ with $\#K = k$ and consider the two (disjoint) societies K and $N \backslash K$ with the respective preference profiles $R' \in \{-1, 0, 1\}^k$ and $R'' \in \{-1, 0, 1\}^{n-k}$ defined as $R'_i = R_i$ for all $i \in K$ and $R''_i = R_i$ for all $i \in N \backslash K$. Note that $n_+(R') = k$ and $n_+(R'') = n_-(R'')$. The former implies $F(R') = 1$ as F is Pareto optimal and the latter implies $F(R'') = 0$ as F is anonymous and neutral. Thus, $|F(R') - F(R'')| \neq 2$, and by WPI we have $F(R' \oplus R'') = F(F(R') \oplus F(R''))$. Noting that $R' \oplus R'' = R$, this is equivalent to $F(R) = F(1, 0)$ which in turn equals 1 as F is PO. One can show (iii) in a similar manner as (ii), which completes the proof. □

Theorem 3.1 gives a characterization result of the majority rule à la May (1952) by replacing the PR condition by PO and WPI. Note that, PO is implicitly included in the result of May as N and PR imply PO. Hence what we do is to drop PR by preserving the implicit PO and using the additional WPI condition.

We wish to remark that, the majority rule satisfies a stronger version of WPI, the same condition defined over any (finite) number of disjoint societies, instead of only two. Thus, we could state Theorem 3.1 by using this (slightly) stronger version of WPI. However, we obtain an impossibility result if we use the (fairly) stronger PI version of WPI. To see this, it suffices to check that the majority rule does not satisfy PI, leading to the following corollary:

Corollary 3.1. *There exists no social welfare function* $F: \cup_{n \in \mathbb{N}} \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ *which satisfies A, N, PO and PI.*

We conclude our discussion by noting that the axioms used in Theorem 3.1 are independent, i.e., none of the three implies the remaining fourth.

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