# 2

# An Introduction to Manipulability

## 2.1 Set Preferences and Manipulability

It has long been known that a voter can sometimes achieve a preferred election result by casting a ballot that misrepresents his or her actual preferences. Over a century ago, C. L. Dodgson referred to a tendency of voters to "adopt a principle of voting which makes it more of a game of skill than a real test of the wishes of the electors" (Black, 1958, p. 232). Dodgson went on to say that in his opinion, it would be "better for elections to be decided according to the wishes of the majority than of those who happen to have most skill at the game" (Black, 1958, p. 233).

The most famous manipulability quote in the history of social choice, however, predates Dodgson by a century or so. It was Jean Charles de Borda's famous reply to a colleague who had pointed out to him how easy it was to manipulate his (Borda's) method of marks (i.e., the Borda count). "My scheme," Borda replied, "is only intended for honest men!" (Black, 1958, p. 182).

Alas, the practice of manipulation today is not restricted to men or women we would consider dishonest. For example, in his 1986 book, *The Art of Political Manipulation*, the late William H. Riker (considered by many to be the intellectual founding father of positive political theory) provides a dozen stories that illustrate the extent to which

> politicians are continually poking and pushing the world to get the results they want. The reason they do this is they believe (and rightly so) that they can change outcomes by their efforts. It is often the case that voting need not have turned out the way it did.

This poking and pushing is the issue we now address.

In a study of manipulation of voting systems, there are two rather distinct types of questions. With the first, one begins with an explicitly given aggregation procedure and attempts to find the ways in which a voter can secure a more favorable election outcome by a unilateral change in his or her ballot. With the second, one starts with an explicit notion of what it means for a voter to prefer one outcome to another and attempts to find all the aggregation procedures (of a certain kind) that are manipulable in this sense. The first kind of question is generally felt to be considerably easier than the second,[7] and it is largely what we pursue in this chapter. The remainder of the book is devoted to the second kind of question.

Throughout this chapter, we assume that ballots are linear. Our starting point is to address the question of how one formalizes the notion of manipulability.

Intuitively, a voting system is manipulable if there exists an election in which some voter can secure an outcome that he or she prefers by unilaterally changing his or her ballot. The ballots of the other voters are held fixed. This corresponds to the assumption that this particular voter has complete knowledge of how everyone else voted (or perhaps better: will vote) and can capitalize on this knowledge to secure a better outcome – better, that is, from his or her point of view – by submitting an insincere ballot. We are, by the way, considering only the kind of manipulation that involves a ballot change by a single voter. Group manipulation, also called *coalitional manipulability*, is discussed in Sections 6.3 and 6.4.

More precisely, a voting rule V is manipulable if there are two profiles $\mathbf{P}$ and $\mathbf{P}'$ and a voter $i$ such that $\mathbf{P}|\mathrm{N} - \{i\} = \mathbf{P}'|\mathrm{N} - \{i\}$ and voter $i$, whose true preferences we take to be $\mathrm{P}_i$, "prefers" $\mathrm{V}(\mathbf{P}')$ to $\mathrm{V}(\mathbf{P})$. If $\mathrm{V}(\mathbf{P}')$ and $\mathrm{V}(\mathbf{P})$ are singletons, say $\mathrm{V}(\mathbf{P}') = \{x\}$ and $\mathrm{V}(\mathbf{P}) = \{y\}$, then there is no doubt what we mean when we say that voter $i$ prefers $\mathrm{V}(\mathbf{P}')$ to $\mathrm{V}(\mathbf{P})$; it simply means that $x\mathrm{P}_i y$. The issue, as we've said earlier, is deciding what it means to say that a voter prefers a set X of alternatives to another set Y.

Almost all the major theorems on manipulability that we present in Chapters 3–8 involve formalizations of set preferences based on one or more of the following ideas:[8]

(1) The case where X and Y are singletons, as we just discussed.
(2) The idea of one set X "weakly dominating" another set Y in the sense that everything in X is at least as good as everything in Y, and something in X is better than something in Y.

[7] I've heard the distinction referred to as low social choice theory versus high social choice theory, but the truth might be that investigators have simply asked (and answered) harder instances of the second question than the first.

[8] There is a vast literature regarding the derivation of subset preferences from preferences over single elements of a set; see, for example, deFinetti (1937), Savage (1954), Fishburn (1986), and Burani and Zwicker (2000). Only some of the resulting ideas have played a prominent role in the study of manipulation of voting systems. Those that have played such a role arise, in one way or another, from an intuition based on the supposition that a single winner will eventually be selected – by a lottery, by a machine, by a person, etc. – from the group initially chosen.

(3) The idea of comparing $\max_i(X, P)$ and/or $\min_i(X, P)$ with $\max_i(Y, P)$ and/or $\min_i(Y, P)$.
(4) The idea of one set X having a higher "expected utility" than another set Y where the calculation is done using some real-valued utility function that represents a voter's preferences and one or more probability functions that provide a measure of the likelihood that a given alternative will ultimately be selected from a given set.

We comment on each of these in turn, and then cull four explicit forms of manipulability from these notions. In the next section of this chapter, we return to our twenty voting rules given in the last chapter and illustrate these different forms of manipulability in that concrete setting.

The first idea above – the case where X and Y are singletons – requires no further comment. The second idea above, that of weak dominance, arises in game theory where one speaks of a strategy for a player as weakly dominating another strategy for that player if the former always yields an outcome at least as good for that player as the latter and sometimes yields an outcome that is strictly better. In point of fact, an election can be thought of as a game in which a strategy for a player (voter) is a choice of ballot, and the outcome of the game is the set of winners in the election.

Thus, we can say that a set X of alternatives is preferred by voter $i$ to a set Y of alternatives in the sense of weak dominance if

$$\forall x \in X\, \forall y \in Y(x R_i y)^9 \quad \text{and} \quad \exists x \in X\, \exists y \in Y(x P_i y).$$

Because ballots are linear, this means that for a set X to weakly dominate a set Y they must have at most one element in common: That is, $\min_i(X, \mathbf{P})\, R_i\, \max_i(Y, \mathbf{P})$ and $X \neq Y$. This, however, allows us to split weak dominance into a "max-version" and a "min-version." In the former, we have

$$\forall x \in X\, \forall y \in Y(x R_i y) \quad \text{and} \quad \max_i(X, \mathbf{P})\, P_i \max_i(Y, \mathbf{P}),$$

and in the latter we have

$$\forall x \in X\, \forall y \in Y(x R_i y) \quad \text{and} \quad \min_i(X, \mathbf{P})\, P_i \min_i(Y, \mathbf{P}).$$

For the third idea, suppose we are given a ballot $P_i$ (in a profile $\mathbf{P}$) that we take to represent the true preferences of voter $i$. A naïve approach yields four ways to use the $\max_i$ and $\min_i$ functions to assert that one set X of alternatives is preferred by voter $i$ to another set Y of alternatives:

(i) $\max_i(X, \mathbf{P})\, P_i \min_i(Y, \mathbf{P})$
(ii) $\min_i(X, \mathbf{P})\, P_i \max_i(Y, \mathbf{P})$

9 Recall that we are assuming that ballots are linear, so $x R_i y$ means $x P_i y$ or $x = y$.

(iii) $\max_i(\mathrm{X}, \mathbf{P})\ \mathrm{P}_i\ \max_i(\mathrm{Y}, \mathbf{P})$
(iv) $\min_i(\mathrm{X}, \mathbf{P})\ \mathrm{P}_i\ \min_i(\mathrm{Y}, \mathbf{P})$

It turns out that (i) and (ii) are not very satisfactory in terms of giving useful notions of manipulability. In particular, (i) is too weak – $\max_i(\mathrm{X}, \mathbf{P})\ \mathrm{P}_i\ \min_i(\mathrm{Y}, \mathbf{P})$ being neither transitive nor irreflexive on sets with more than one element, and (ii) is strong enough so as to be somewhat redundant – manipulations resulting in $\min_i(\mathrm{X}, \mathbf{P})\ \mathrm{P}_i\ \max_i(\mathrm{Y}, \mathbf{P})$ most commonly achievable only when the sets X and Y can, in fact, be taken to be singletons.

However, there are some reasonably good intuitions behind the use of (iii) and (iv) in manipulability investigations. For example, let's assume that, when the dust settles, society will need to have a single winner, and that this single winner will be selected in some way (randomly, by some committee, etc.) from those tied for the win according to our voting rule.

Now, if a voter is sufficiently optimistic, and if he or she ranks $a$ over $b$ over $c$ over $d$, then he or she will prefer an election outcome of $\{a, d\}$ to an election outcome of $\{b, c\}$. This is because he or she will assume – optimistically – that $a$ (his or her top choice overall) will result from an election outcome of $\{a, d\}$, while $b$ (his or her second choice overall) will result from an election outcome of $\{b, c\}$. In general, a sufficiently optimistic voter will compare two election outcomes (that is, two sets of alternatives) by asking which has a larger max according to the voter's true preference ranking of the alternatives – that is, by using (iii).

On the other hand, if a voter is sufficiently pessimistic, and if he or she ranks $a$ over $b$ over $c$ over $d$, then he or she will prefer an election outcome of $\{b, c\}$ to an election outcome of $\{a, d\}$. This is because he or she will assume – pessimistically – that $d$ (his or her worst choice overall) will result from an election outcome of $\{a, d\}$, while $c$ (his or her third choice overall) will result from an election outcome of $\{b, c\}$. In general, a sufficiently pessimistic voter will compare two election outcomes (that is, two sets of alternatives) by asking which has a larger min according to his or her true preference ranking of the alternatives (that is, by using (iv)).

Another way to view this notion of manipulation by an optimist or a pessimist is to return to our example in Chapter 1 wherein we had ten faculty members in an academic department trying collectively to choose from among five candidates for a position in the department. Any of the voting rules that come to mind (plurality, Hare, Borda, etc.) will produce ties upon occasion, and one option is to let the dean break any ties that arise. Here, optimism and pessimism need not be any kind of general state of mind. An optimist is simply a department member who feels that the dean shares his or her values (e.g., the relative importance attached to effective teaching versus a prominent research profile); a pessimist is one who feels just the opposite.

Finally, our fourth notion is based on the idea that one might want to say that a voter prefers a set X of alternatives to a set Y of alternatives if his or her "expected utility" from X is greater than his or her "expected utility" from Y. Although we put expected utility in quotes for good reason, the intuition here is quite clear; the expected utility of a set X of alternatives to a voter should be the sum, over all $x \in$ X, of the product of the following two numbers:

(1) The "value" or "utility" of alternative $x$ to that voter.
(2) The probability with which that voter sees alternative $x$ emerging as the eventual winner from the set X.

For this kind of arithmetic calculation to make sense, we want the "utility" referred to in (1) to be a number. This is achieved if each voter has a so-called *utility function u* mapping the set A of alternatives to the set $\Re$ of real numbers (denoted $u$: A $\rightarrow \Re$) *that represents his or her preferences* (for individual alternatives) in the sense that for every $x$, $y \in$ A, $x\mathrm{P}_i y$ iff $u(x) > u(y)$. Additionally, (2) requires that each voter has, for every set X of alternatives, a *probability function on X*, that is, a function $p$: X $\rightarrow$ [0, 1] such that $\sum\{p(x) : x \in \mathrm{X}\} = 1$. Here again there are two natural ways in which $p$ might arise:

(1) The probability function $p$ might depend on the particular voter and his or her knowledge or suppositions about how ties will ultimately be resolved. In this case, the nature of $p$ might vary from voter to voter and from set to set.
(2) The probability function $p$ might be determined by the procedure itself. For example, if the procedure were to specify that ties must be broken randomly, then we would have $p(x) = 1/|\mathrm{X}|$ for every $x \in$ X.

There are six expected-utility notions of manipulability arising from the situation described in (1), and all of these have combinatorial equivalents that make use of the $\min_i$ and $\max_i$ functions (sometimes in ways that are quite different from what we had above for optimists and pessimists). This material is presented in Section 4.4.

For the moment, however, we want to focus on (2) and the notion of manipulability arising from saying that a set X of alternatives is preferred (or perhaps better, can be preferred) to a set Y of alternatives by voter $i$ if there exists a utility function $u$ representing $\mathrm{P}_i$ such that, if $p(x) = 1/|\mathrm{X}|$ for every $x \in$ X, and

$p(y) = 1/|\mathrm{Y}|$ for every $y \in \mathrm{Y}$, then

$$\sum\{p(x) \cdot u(x) : x \in \mathrm{X}\} > \sum\{p(y) \cdot u(y) : y \in \mathrm{Y}\}^{10}.$$

This notion was introduced by Feldman, and we illustrate it in the course of proving Theorem 2.3.1 in this chapter. Notationally, if $u$ is a utility function representing $\mathrm{P}_i$, then we let

$$\mathrm{E}_{u,i}(\mathrm{X}) = \sum\{u(x) : x \in \mathrm{X}\}/|\mathrm{X}|.$$

Thus, in the special case where $p(x) = 1/|\mathrm{X}|$ for every $x \in \mathrm{X}$, the "expected utility of X" might also be called the "mean (or average) utility of X."

This completes our discussion of the four fundamental ideas underlying the sense in which a voter might prefer one set of alternatives to another. These ideas, in turn, give rise to the four primary notions of manipulability that we need to analyze the specific voting rules from the last chapter and to summarize, in Section 2.3, some of the main results presented in other chapters. These four notions of manipulability (with comments to follow that allow for slightly finer distinctions) are collected in the following definition.[11]

**Definition 2.1.1.** In the context of linear ballots, a voting rule is:

(1) *single-winner manipulable* if there exist profiles $\mathbf{P}$ and $\mathbf{P}'$ and a voter $i$ such that $\mathbf{P}|\mathrm{N} - \{i\} = \mathbf{P}'|\mathrm{N} - \{i\}$ and voter $i$, whose true preferences we take to be given by his or her ballot in $\mathbf{P}$, prefers the election outcome X from $\mathbf{P}'$ to the election outcome Y from $\mathbf{P}$ in the following sense:

$$\mathrm{X} = \{x\} \quad \text{and} \quad \mathrm{Y} = \{y\} \quad \text{and} \quad x\mathrm{P}_i y.$$

(2) *weak-dominance manipulable* if there exist profiles $\mathbf{P}$ and $\mathbf{P}'$ and a voter $i$ such that $\mathbf{P}|\mathrm{N} - \{i\} = \mathbf{P}'|\mathrm{N} - \{i\}$ and voter $i$, whose true preferences we take to be given by his or her ballot in $\mathbf{P}$, prefers the election outcome X from $\mathbf{P}'$ to the election outcome Y from $\mathbf{P}$ in the following sense:

$$\forall x \in \mathrm{X}\, \forall y \in \mathrm{Y}(x\mathrm{R}_i y) \quad \text{and} \quad \exists x \in \mathrm{X}\, \exists y \in \mathrm{Y}(x\mathrm{P}_i y).$$

(3a) *manipulable by optimists* if there exist profiles $\mathbf{P}$ and $\mathbf{P}'$ and a voter $i$ such that $\mathbf{P}|\mathrm{N} - \{i\} = \mathbf{P}'|\mathrm{N} - \{i\}$ and voter $i$, whose true preferences we take

[10] If we demanded that the utility function take on only positive real values, then the notion of X being preferred to Y in the sense of expected utility would be unchanged. See Exercise 7.

[11] Attempts to organize the various kinds of manipulability that suggest themselves date back at least to Gärdenfors (1979). Our own experience with this began with several undergraduate theses we supervised, including that of Ryan Kindl and Matthew Gendron. Related material can be found in Bartholdi and Orlin (1991) and Smith (1999).

to be given by his or her ballot in **P**, prefers the election outcome X from **P**′ to the election outcome Y from **P** in the following sense:

$$\max_i(X, \mathbf{P})\, P_i\, \max_i(Y, \mathbf{P})$$

(3b) *manipulable by pessimists* if there exist profiles **P** and **P**′ and a voter $i$ such that $\mathbf{P}|N - \{i\} = \mathbf{P}'|N - \{i\}$ and voter $i$, whose true preferences we take to be given by his or her ballot in **P**, prefers the election outcome X from **P**′ to the election outcome Y from **P** in the following sense:

$$\min_i(X, \mathbf{P})\, P_i\, \min_i(Y, \mathbf{P})$$

(4) *expected-utility manipulable* if there exist profiles **P** and **P**′ and a voter $i$ such that $\mathbf{P}|N - \{i\} = \mathbf{P}'|N - \{i\}$ and voter $i$, whose true preferences we take to be given by his or her ballot in **P**, prefers the election outcome X from **P**′ to the election outcome Y from **P** in the following sense: There exists a utility function $u$ representing $P_i$ such that, if $p(x) = 1/|X|$ for every $x \in X$, and $p(y) = 1/|Y|$ for every $y \in Y$, then $\sum\{p(x) \cdot u(x) : x \in X\} > \sum\{p(y) \cdot u(y) : y \in Y\}$; i.e., $E_{u,i}(X) > E_{u,i}(Y)$.

Exercise 2 at the end of the chapter asks for verification that the conditions imposed on X and Y in the definition of weak-dominance manipulability hold iff at least one of the following is true:

(i) $X = \{x\}$ and $Y = \{y\}$ and $xP_iy$
(ii) $\max_i(X, \mathbf{P})\, P_i\, \min_i(X, \mathbf{P})\, R_i\, \max_i(Y, \mathbf{P})$
(iii) $\min_i(X, \mathbf{P})\, R_i\, \max_i(Y, \mathbf{P})\, P_i\, \min_i(Y, \mathbf{P})$

In case (ii) we will use the phrase "max-weak-dominance manipulable," and in case (iii) the phrase "min-weak-dominance manipulable." Exercise 3 asks for a proof that single-winner manipulability implies weak-dominance manipulability, weak-dominance manipulability implies manipulability by optimists or pessimists, and manipulability by optimists or pessimists implies expected-utility manipulability.

We show in Section 4.4 that the set preference notion used in the definition of manipulation by optimists or pessimists is equivalent to some expected-utility notions that differ from what is given above in a couple of important ways. Conversely, the version of expected-utility manipulation given above has a very nice combinatorial equivalent, but not in terms of the $\max_i$ and $\min_i$ functions. Roughly, it says that a set X is preferred to a set Y if there is some alternative $z$ such that voter $i$ has a larger fraction of X than Y at or above $z$ on his or her ballot (see Exercise 4). But let us now turn to the twenty voting rules introduced earlier and see how they stack up in terms of inducing honesty.

## 2.2 Specific Examples of Manipulation

The four kinds of manipulability that we have at hand, from strongest to weakest, are single-winner manipulability, weak dominance manipulability, manipulability by optimists and/or pessimists, and expected-utility manipulation. Our first result illustrates these varying levels of manipulability with several of the voting rules presented in the last chapter. Notice that the procedures in (i)–(iv) below are anonymous, neutral, monotone, and non-imposed.

**Theorem 2.2.1.**[12]

*(i) For* A = {a, b, c, d}, *the Borda count for* $(A, 4)$ *is single-winner manipulable.*

*(ii) For* A = {a, b, c}, *the plurality rule for* $(A, 4)$ *is weak-dominance manipulable. However, it is never single-winner manipulable.*

*(iii) For* A = {a, b, c}, *the Condorcet rule for* $(A, 3)$ *is manipulable by both optimists and pessimists. However, it is never weak-dominance manipulable.*

*(iv) For* A = {a, b, c}, *the nomination-with-second rule for* $(A, 4)$ *is manipulable by optimists (but never by pessimists if* $|A| < n$*), and the near-unanimity rule for* $(A, 3)$ *is manipulable by pessimists (but never by optimists).*

*(v) For* A = {a, b, c}, *the Pareto rule for* $(A, 3)$ *is expected-utility manipulable. However, it is never manipulable by optimists or pessimists.*

*(vi) Dictatorships and duumvirates are never expected-utility manipulable.*

Let us make a couple of comments before turning to the proof. First, parts (iv) and (vi) of the theorem involve procedures that few would advocate for real-world adoption, but the results are important for the theory in other chapters. Second, of the four "real-world voting systems" in the theorem, we have that the Borda count is (in one sense, at least) most manipulable, followed by the plurality rule, Condorcet's rule, and the Pareto rule in that order.

Each time we need to show that a voting rule is manipulable in some sense, we produce a positive integer $n$, a set A of alternatives, and two linear (A, $n$)-profiles **P** and **P′** that provide an instance of manipulation by voter 1 (who is at the far left) when we regard his or her true preferences to be given by his or her ballot in **P**. We leave the verification that the election winners are what we say they are to the exercises. For notation, we let $F(x, \mathbf{P}) = |\{i \in N\colon \mathrm{Top}_i(\mathbf{P}) = x\}|$, where we think of "F" as standing for "first."

[12] For a finer analysis of the manipulability of these procedures, see the exercises at the end of the chapter. For example, Exercise 8 gives an example of a voting rule that is single-winner manipulable for (A, $n$) when $|A| = 3$ and $n = 3$, and Exercise 10 asks for a proof that the Borda count is not single-winner manipulable when $|A| = 3$.

*Proof:* For (i), let A $= \{a, b, c, d\}$, let $n = 4$, and consider the following profiles **P** and **P**$'$:

| **P** | | | | **P**$'$ | | | |
|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $d$ | $c$ | $b$ | $b$ | $d$ | $c$ |
| $b$ | $d$ | $c$ | $a$ | $a$ | $d$ | $c$ | $a$ |
| $c$ | $c$ | $a$ | $b$ | $d$ | $c$ | $a$ | $b$ |
| $d$ | $a$ | $b$ | $d$ | $c$ | $a$ | $b$ | $d$ |

If V is the Borda count, then V(**P**) $= \{c\}$ and V(**P**$'$) $= \{b\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second choice.

For (ii), let A $= \{a, b, c\}$, let $n = 4$, and consider the following profiles **P** and **P**$'$:

| **P** | | | | **P**$'$ | | | |
|---|---|---|---|---|---|---|---|
| $a$ | $c$ | $c$ | $b$ | $b$ | $c$ | $c$ | $b$ |
| $b$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $b$ | $b$ | $c$ | $c$ | $b$ | $b$ | $c$ |

If V is the plurality rule, then V(**P**) $= \{c\}$ and V(**P**$'$) $= \{b, c\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second and third choices. This shows that the plurality rule is max weak-dominance manipulable. The plurality rule can also be min weak-dominance manipulable, depending on the number of alternatives and the number of voters; see Exercise 13.

For the second claim, we will show that, with the plurality rule, no voter can ever simultaneously improve the min and the max of the set of winners. From this it follows that the plurality rule is not single-winner manipulable. In fact, we'll show that if **P**|N $- \{i\} =$ **P**$'$|N $- \{i\}$, then either V(**P**) $\subseteq$ V(**P**$'$) or V(**P**$'$) $\subseteq$ V(**P**).

Assume for contradiction that **P**|N $- \{i\} =$ **P**$'$|N $- \{i\}$ and that we can choose $x \in$ V(**P**) $-$ V(**P**$'$) and $y \in$ V(**P**$'$) $-$ V(**P**). Without loss of generality, assume that $y\mathrm{P}_i x$, and let F($x$, **P**) $= k$. Notice that because $x$ is not at the top of voter $i$'s ballot, we also have F($x$, **P**$'$) $\geq k$. Because $x \in$ V(**P**) and $y \notin$ V(**P**), we know that F($y$, **P**) $\leq k - 1$. But now, because **P**|N $- \{i\} =$ **P**$'$|N $- \{i\}$, we know that F(y, **P**$'$) $\leq k$. It now follows that because $y \in$ V(**P**$'$), we have $x \in$ V(**P**$'$) because it also has at least $k$ first-place votes in **P**$'$, and this is the desired contradiction.

For (iii), let A = $\{a, b, c\}$, let $n = 3$, and consider the following profiles **P** and **P′**:

| **P** | | | **P′** | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $a$ | $b$ | $c$ |
| $c$ | $c$ | $a$ | $b$ | $c$ | $a$ |
| $b$ | $a$ | $b$ | $c$ | $a$ | $b$ |

If V is the Condorcet rule, then V(**P**) = $\{c\}$ and V(**P′**) = $\{a, b, c\}$, so voter 1 has improved the max of the election outcome from being his or her second choice to being his or her first choice. This shows that the Condorcet rule is manipulable by optimists. For the proof that it is also manipulable by pessimists, see Exercise 15.

For the second claim, assume that **P**|N − $\{i\}$ = **P′**|N − $\{i\}$, that V(**P**) = Y, that V(**P′**) = X, and that X weakly dominates Y with respect to $P_i$, which we take to be voter $i$'s true preferences. Notice first that if Y and X are singletons, say Y = $\{y\}$ and X = $\{x\}$, then we must have $xP_iy$ for X to weakly dominate Y. But this is impossible, because then $y$ would still defeat $x$ one-on-one after voter $i$'s ballot change.

It thus follows that one of X and Y is a singleton, and the other is the whole set A. If X = $\{x\}$, then $x$ must be at the top of voter $i$'s ballot in **P** in order to have X dominate Y. But V(**P**) = A, and so $x$ was not a Condorcet winner. Clearly, no change in voter $i$'s ballot can covert his or her top choice from not being a Condorcet winner to being a Condorcet winner. Similarly, if Y = $\{y\}$, then $y$ must have been at the bottom of voter $i$'s ballot in **P** in order to have X dominate Y. But then $y$ will remain a Condorcet winner no matter how voter $i$ changes his or her ballot. With the Condorcet rule, it also turns out that a voter can never simultaneously improve the max and the min of the set of winners; see Exercise 24.

For (iv), let A = $\{a, b, c\}$, let $n = 4$, and consider the following profiles **P** and **P′** (which are the same as in (ii)):

| **P** | | | | **P′** | | | |
|---|---|---|---|---|---|---|---|
| $a$ | $c$ | $c$ | $b$ | $b$ | $c$ | $c$ | $b$ |
| $b$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $b$ | $b$ | $c$ | $c$ | $b$ | $b$ | $c$ |

If V is the nomination-with-second rule, then V(**P**) = $\{c\}$ and V(**P′**) = $\{b, c\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second and third choices. This shows that the nomination-with-second rule is manipulable by optimists (in fact, max weak-dominance manipulable).

For the second claim regarding the nomination-with-second rule, we have $|A| < n$, so there is always at least one alternative with at least two first-place votes. So the min for voter $i$ is either his or her top choice, or it is at the top of two other voter's ballots, in which case he or she can never make it a loser.

For the near-unanimity rule, let $A = \{a, b, c\}$, let $n = 3$, and consider the following profiles **P** and **P′**:

| | **P** | | | **P′** | |
|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $b$ | $b$ | $c$ |
| $b$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $c$ | $b$ | $c$ | $c$ | $b$ |

If V is the near-unanimity rule, then $V(\mathbf{P}) = \{a, b, c\}$ and $V(\mathbf{P}') = \{b\}$, so voter 1 has improved the min of the election outcome from being his or her third choice to being his or her second choice. This shows that the near-unanimity rule is manipulable by pessimists.

To show that the near-unanimity rule is not manipulable by optimists, notice that the max for voter $i$ is either his top choice, or it is some single alternative at the top of every other voter's ballots, in which case he or she can never change the election outcome.

For (v), let $A = \{a, b, c\}$, let $n = 3$, and consider the following profiles **P** and **P′**:

| | **P** | | | **P′** | |
|---|---|---|---|---|---|
| $a$ | $a$ | $c$ | $a$ | $a$ | $c$ |
| $b$ | $c$ | $b$ | $c$ | $c$ | $b$ |
| $c$ | $b$ | $a$ | $b$ | $b$ | $a$ |

If V is the Pareto rule then $V(\mathbf{P}) = \{a, b, c\}$ and $V(\mathbf{P}') = \{a, c\}$. Now let $u$ be any utility function representing voter 1's preferences in **P** with the average of $u(a)$ and $u(c)$ greater than $u(b)$. For definiteness, let's take $u(a) = 18$, $u(b) = 9$ and $u(c) = 6$. Then the expected utility from $\{a, b, c\}$ is

$$(1/3) \cdot 18 + (1/3) \cdot 9 + (1/3) \cdot 6 = 6 + 3 + 2 = 11,$$

and the expected utility from $\{a, c\}$ is

$$(1/2) \cdot 18 + (1/2) \cdot 6 = 9 + 3 = 12.$$

This shows that the Pareto rule is expected-utility manipulable.

It's easy to see that the Pareto rule can't be manipulated by optimists, since $\text{top}_i(\mathbf{P}) \in \text{V}(\mathbf{P})$ for every $i \in \text{N}$. To see that it can't be manipulated by pessimists takes a little more work. Let $y = \min_i(\text{Y})$ where $\text{Y} = \text{V}(\mathbf{P})$. If $y$ is at the bottom of voter $i$'s ballot, then $y$ will certainly remain a winner no matter how voter $i$ changes his or her ballot. So we can assume that there are alternatives $z_1, \ldots, z_k$ that voter $i$ has below $y$ on his or her ballot and that are non-winners. For voter $i$ to make $y$ a loser, he or she must do it by raising at least one of the $z$'s over $y$ so that, when this is done, every voter will have that $z$ over $y$.

Choose $z_1$ to be such that voter $i$ has $z_1$ below $y$, but every other voter has $z_1$ above $y$. Because $z_1$ is a non-winner (being below $y$ on voter $i$'s ballot), we can choose some $z_2$ such that every voter has $z_2$ over $z_1$, and, in particular, every voter except voter $i$ has $z_2$ above $y$. But because $y$ is a winner, voter $i$ must have $z_2$ below $y$. But now $z_2$ has the same properties as did $z_1$, and we can continue to produce $z_3$, $z_4$, etc., forever.

Finally the statement in (vi) is trivial, and this completes the proof of Theorem 2.2.1. □

In addition to the Borda count, eight of our other voting rules are also single-winner manipulable. The following theorem gives seven; finding the eighth is left to the reader (Exercise 27).

**Theorem 2.2.2.** *For each of the following, there exists an* n $\geq$ *1 and a set A of alternatives such that the voting rule for* (A, n) *is single-winner manipulable:*

*(1) The plurality runoff rule*
*(2) The weak Condorcet rule*
*(3) Copeland's rule*
*(4) The sequential pairwise rule*
*(5) The Hare system*
*(6) The Coombs rule*
*(7) The iterated plurality rule*

*Proof:* In each of the seven cases, we again produce a positive integer $n$, a set A of alternatives, and two linear (A, $n$)-profiles $\mathbf{P}$ and $\mathbf{P}'$ that provide an instance of single-winner manipulation by voter 1 (who is at the far left) when we regard his or her true preferences to be given by his or her ballot in **P**.

(1) The plurality runoff rule: Let A = $\{a, b, c\}$, let $n = 5$, and consider the following profiles **P** and **P′**:

| **P** | | | | | **P′** | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $c$ | $c$ | $b$ | $b$ | $a$ | $c$ | $c$ | $b$ |
| $b$ | $b$ | $a$ | $a$ | $c$ | $a$ | $b$ | $a$ | $a$ | $c$ |
| $c$ | $c$ | $b$ | $b$ | $a$ | $c$ | $c$ | $b$ | $b$ | $a$ |

If V is the plurality runoff rule, then V(**P**) = $\{c\}$ and V(**P′**) = $\{b\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second choice.

(2) The weak Condorcet rule: Let A = $\{a, b, c, d\}$, let $n = 4$, and consider the following profiles **P** and **P′**:

| **P** | | | | **P′** | | | |
|---|---|---|---|---|---|---|---|
| $a$ | $c$ | $b$ | $d$ | $b$ | $c$ | $b$ | $d$ |
| $b$ | $a$ | $d$ | $c$ | $a$ | $a$ | $d$ | $c$ |
| $c$ | $b$ | $c$ | $a$ | $d$ | $b$ | $c$ | $a$ |
| $d$ | $d$ | $a$ | $b$ | $c$ | $d$ | $a$ | $b$ |

If V is the weak Condorcet rule, then V(**P**) = $\{c\}$ and V(**P′**) = $\{b\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second choice.

(3) Copeland's rule: Let A = $\{a, b, c, d, e\}$, let $n = 4$, and consider the following profiles **P** and **P′**:

| **P** | | | | **P′** | | | |
|---|---|---|---|---|---|---|---|
| $a$ | $c$ | $a$ | $d$ | $c$ | $c$ | $a$ | $d$ |
| $b$ | $e$ | $e$ | $b$ | $a$ | $e$ | $e$ | $b$ |
| $c$ | $d$ | $d$ | $e$ | $b$ | $d$ | $d$ | $e$ |
| $d$ | $b$ | $c$ | $c$ | $e$ | $b$ | $c$ | $c$ |
| $e$ | $a$ | $b$ | $a$ | $d$ | $a$ | $b$ | $a$ |

If V is Copeland's rule, then V(**P**) = $\{d\}$ and V(**P′**) = $\{c\}$, so voter 1 has improved the election outcome from being his or her fourth choice to being his or her third choice.

(4) The sequential pairwise rule: Let A = $\{a, b, c\}$, let $n = 3$, and consider the following profiles **P** and **P**′:

| **P** | | | **P**′ | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $b$ | $b$ | $c$ |
| $b$ | $c$ | $a$ | $a$ | $c$ | $a$ |
| $c$ | $a$ | $b$ | $c$ | $a$ | $b$ |

If V is the sequential pairwise rule with the ordering of the alternatives being $abc$, then V(**P**) = $\{c\}$ and V(**P**′) = $\{b\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second choice.

(5) The Hare system: Let A = $\{a, b, c, d\}$, let $n = 5$, and consider the following profiles **P** and **P**′:

| **P** | | | | | **P**′ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $c$ | $d$ | $b$ | $b$ | $c$ | $c$ | $d$ |
| $b$ | $a$ | $b$ | $b$ | $b$ | $a$ | $a$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $a$ | $a$ | $c$ | $c$ | $c$ | $a$ | $a$ | $c$ |
| $d$ | $d$ | $d$ | $d$ | $a$ | $d$ | $d$ | $d$ | $d$ | $a$ |

If V is the Hare system, then V(**P**) = $\{c\}$ and V(**P**′) = $\{b\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her second choice.

(6) The Coombs rule: Let A = $\{a, b, c\}$, let $n = 5$, and consider the following profiles **P** and **P**′:

| **P** | | | | | **P**′ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $b$ | $a$ | $a$ | $a$ | $b$ | $b$ | $a$ | $a$ |
| $b$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ |
| $c$ | $a$ | $a$ | $b$ | $b$ | $b$ | $a$ | $a$ | $b$ | $b$ |

If V is the Coombs rule, then V(**P**) = $\{c\}$ and V(**P**′) = $\{a\}$, so voter 1 has improved the election outcome from being his or her third choice to being his or her first choice.

(7) The iterated plurality rule: See Exercise 26.

This completes the proof of Theorem 2.2.2. □

The manipulability of the last four of our voting rules – the unanimity and nomination rules and the oligarchies and triumvirates – is left to the reader (See Exercises 29–32).

## 2.3 Summary of the Main Results

All four notions of manipulability – single-winner manipulation, weak dominance manipulation, manipulation by optimists or pessimists, and expected-utility manipulation – arise in one or more of the general theorems that we elsewhere present in this book. These results all assert that a large class of voting systems are susceptible to manipulation in that particular sense. By way of summary, we give the statement (but not the proof) of some of those theorems here.

Single-winner manipulation is a very strong notion indeed, and it is precisely the notion addressed by the seminal Gibbard–Satterthwaite theorem. Unfortunately, the class of voting rules that it identifies as being susceptible to this kind of manipulation – that is, the non-imposed, resolute voting rules that are not dictatorships – omits, for example, nineteen of the twenty voting rules that we presented in the last chapter, at least for some choices of A and $n$.

One can extend the applicability of the Gibbard–Satterthwaite theorem in two different but related ways: One can change the voting rules so that they become resolute, or one can change the theorem so that it applies to non-resolute procedures. Both approaches, it turns out, involve the same idea – using a linear ordering of the alternatives (perhaps one of the ballots, perhaps a fixed "absentee ballot") as a tie-breaker. Unfortunately, using a random device as a tie-breaker leaves the Gibbard–Satterthwaite theorem inapplicable, because the resulting procedure is not a function.

We record the Gibbard–Satterthwaite theorem here in both its resolute and non-resolute forms. The resolute version reappears (with proof) as Theorems 3.1.2 and Corollary 3.1.12. For the equivalence of the non-resolute version, see Corollary 3.1.13.

**Theorem 2.3.1.** *For every set A of three or more alternatives and every* n $\geq 1$, *every resolute voting rule for* $(A, \mathrm{n})$ *that is non-imposed and not a dictatorship is single-winner manipulable.*

*Equivalently, with the same assumptions on A and* n, *a (not necessarily resolute) voting rule that is non-imposed and not a dictatorship is manipulable in the sense that there exist profiles* **P** *and* **P**′ *and a voter* i *such that* $\mathbf{P}/N - \{\mathrm{i}\} = \mathbf{P}'/N - \{\mathrm{i}\}$ *and voter* i, *whose true preferences we take to be given by his or* *her ballot in **P**, prefers the election outcome X from **P**′ to the election outcome Y from **P** in the following sense:*

$$max_i(X - Y, \mathbf{P})P_i\, min_i(Y, \mathbf{P}) \quad or \quad max_i(X, \mathbf{P})P_i\, min_i(Y - X, \mathbf{P}).$$

If we had simply said "$\max_i(X, \mathbf{P})\ P_i\ \min_i(Y, \mathbf{P})$," then the corresponding not-necessarily-resolute version above would still have been strong enough to imply the resolute version of the Gibbard–Satterthwaite theorem. However, the relation given by $\max_i(X, \mathbf{P})\ P_i\ \min_i(Y, \mathbf{P})$ is reflexive on non-singleton sets, so *every* non-resolute social choice function is manipulable in this sense by simply choosing an election with more than one winner and then having a voter (any voter) make absolutely no change at all in his or her ballot.

Limitations such as these seem to be the price we pay for dealing with a notion of manipulability as strong as single-winner manipulability. The plurality rule, after all, is not manipulable in this sense. (Exercise 33 asks the reader to reconcile this fact with the opening paragraph of the preface.) Our second notion, weak-dominance manipulability, is also quite strong, but there are several results in the social choice function context showing that it is achievable for a large class of procedures.

We need a couple of definitions pertaining to a social choice function V. First, V is *quasitransitive* if, for every profile **P** and for every triple $\{x, y, z\} \in [A]^3$, if $V(\mathbf{P})(\{x, y\}) = \{x\}$ and $V(\mathbf{P})(\{y, z\}) = \{y\}$, then $V(\mathbf{P})(\{x, z\}) = \{x\}$. Second, V is *pairwise non-imposed* if for every $(x, y) \in A \times A$, there exists a profile **P** such that $V(\mathbf{P})(\{x, y\}) = \{x\}$. Third, V is a *pairwise oligarchy* if there exists a set O of voters such that for every profile **P** and for every $\{x, y\} \in [A]^2$,

$$V(\mathbf{P})(\{x, y\}) = \begin{cases} \{x\} & \text{if } \forall i \in O\ xP_i y \\ \{y\} & \text{if } \forall i \in O\ yP_i x \\ \{x, y\} & \text{otherwise} \end{cases}$$

The following result (Barberá, 1977a and Kelly, 1977) reappears (with proof) as Theorem 5.1.14.

**Theorem 2.3.2.** *For every set A of three or more alternatives and every* $n \geq 1$, *every social choice function for* $(A, n)$ *that is quasitransitive, pairwise non-imposed, and not a pairwise oligarchy is weak-dominance manipulable on some two-element agenda.*

In order to state the main result regarding manipulation by optimists or pessimists, we need one definition: A voter $i$ is said to be a *nominator* for a voting rule V if $\text{top}_i(\mathbf{P}) \in V(\mathbf{P})$ for every profile **P**. Thus, for example, every voter is a nominator for both the Pareto rule and the omninomination rule. The following result is due to Duggan and Schwartz (1993 and 2000) and reappears (with proof) as Theorem 4.1.2.

**Theorem 2.3.3.** *For every set A of three or more alternatives and every* n $\geq 1$, *every voting rule for* $(A, \mathrm{n})$ *that is non-imposed and has no nominators is manipulable by either optimists or pessimists.*

It follows from the Duggan–Schwartz theorem that, among all voting rules that are anonymous, non-imposed, and not manipulable by optimists or by pessimists, the omninomination rule V is the most discriminating in the sense that, for every profile **P**, we have $\mathrm{V}(\mathbf{P}) \subseteq \mathrm{V}'(\mathbf{P})$ for every member V′ of the class.

Finally, for expected-utility manipulation we have the following theorem of Feldman (1979a). It reappears (with proof) as Theorem 4.3.2.

**Theorem 2.3.4.** *For every set A of three or more alternatives and every* n $\geq 1$, *every voting rule for* $(A, \mathrm{n})$ *that is non-imposed and neither a dictatorship nor a duumvirate is expected-utility manipulable.*

This completes our summary of the main results to follow. We conclude this chapter with an isolated look at a very different kind of manipulability.

## 2.4 Agenda Manipulability and Transitive Rationality

Manipulation really comes in two flavors, explicitly articulated by Riker (1982, p. 137).

> If we assume that society discourages the concentration of power [thus ruling out dictatorships, for example], then at least two methods of manipulation are always available, no matter what method of voting is used: First, those in control of procedures can manipulate the agenda (by, for example, restricting alternatives or by arranging the order in which they are brought up). Second, those not in control can still manipulate outcomes by false revelation of values.

Ballot manipulation is what we have been doing, and what the rest of this book is about. But in this brief section, we give an elaboration of Kelly's reference (Kelly, 1978, p. 79) to a weakened version of transitive rationality known as path independence as "another kind of strategy-proofness, dealing not with manipulation of preferences, but with manipulation of agendas." First, we need a definition of what it means to say that a social choice function is subject to agenda-manipulation.

With a social choice function, it certainly makes sense to ask if a voter's thwarted preference for $x$ versus $y$ – thwarted in that $y$ actually wins over $x$ as things now stand – might be overcome if he or she could change (or could have changed) the agenda by adding some new alternatives to it or by deleting some other alternatives (other than $x$ and $y$, that is) from it. Fishburn (1973, pp. 7 and 8) says that "we may consider a maneuver in which an alternative is legally placed in nomination not because its sponsors think it has any chance of being elected but because they feel that its introduction will increase the chance of the election of their favored alternative." These considerations yield the following.

**Definition 2.4.1.** A resolute social choice function V is *agenda manipulable* if there exists a profile **P**, two agendas $v$ and $v'$, two alternatives $x$ and $y$ in $v \cap v'$, and a voter $i$ such that $x\mathrm{P}_i y$, $\mathrm{V}(\mathbf{P})(v) = y$, and $\mathrm{V}(\mathbf{P})(v') = x$. If V is not agenda manipulable, then V is *agenda non-manipulable*.

Intuitively, we are thinking of $v$ as having been the original agenda, and voter $i$'s strict preference for $x$ versus $y$ being originally thwarted by $y$'s ability to win over $x$ with the agenda $v$. However, if voter $i$ were to have agenda-setting power, he or she could – while retaining $y$ in the agenda – switch to $v'$ and obtain $x$ as the winner.

Riker (1986, p. 148) gives a real-world example of exactly this kind of agenda manipulation. He describes how Thomas B. Reed, a Republican member of congress in the late 1800s frustrated his opponents' attempt to expand the agenda from {yes, no} to {yes, no, abstain}. Had his opponents succeeded, the outcome of the election discussed there might well have changed.

Regarding agenda manipulation, we have the following observation.

**Theorem 2.4.2.** *In the linear-ballot context, a resolute social choice function is agenda non-manipulable iff it satisfies Arrow's condition of transitive rationality.*

*Proof:* Assume first that V is a resolute social choice function that satisfies transitive rationality, and let V′ be the social welfare function that gives rise to V. Let **P** be a profile and assume that $\mathrm{V}'(\mathbf{P}) = \mathrm{R}$. Now, if we have two agendas $v$ and $v'$ with $x$ and $y$ in both, then $\mathrm{V}(\mathbf{P})(v) = y$ implies that $y$ is ranked strictly higher than $x$ according to the relation R. Hence, we can't have $\mathrm{V}(\mathbf{P})(v') = x$, and this shows that V is agenda non-manipulable.

Conversely, assume that V is a resolute social choice function that is agenda non-manipulable. We will produce the social welfare function V′ from which V arises. Let **P** be any profile, and set $\mathrm{V}'(\mathbf{P}) = \mathrm{R}$, where R is defined as follows:

$$\text{For } x, y \in \mathrm{A},\ x\mathrm{R}y \text{ iff } \mathrm{V}(\mathbf{P})(\{x, y\}) = x.$$

For the rest of the proof, the profile **P** is fixed, so we will write "V($v$)" instead of "V(**P**)($v$)."

Claim 1. R is transitive.

Proof. Suppose that $x\text{R}y$ and $y\text{R}z$, and assume, for contradiction, that $x\text{R}z$ fails. Thus, $\text{V}(\{x, y\}) = x$ and $\text{V}(\{y, z\}) = y$, but $\text{V}(\{x, z\}) \neq x$. Because V is resolute, this means that $\text{V}(\{x, z\}) = z$. Again, because V is resolute, we know that $\text{V}(\{x, y, z\})$ is either $x$, $y$, or $z$. Without loss of generality, assume that $\text{V}(\{x, y, z\}) = x$.[13]

Fix a voter $i$, and suppose first that $x\text{P}_i z$ (that is, suppose that voter $i$ ranks $x$ over $z$ on his or her ballot). Then, if $v = \{x, z\}$, voter $i$'s preference for $x$ over $z$ is initially thwarted by the fact that $\text{V}(\{x, z\}) = z$. But if voter $i$ has agenda-setting power, he or she can add $y$ to the agenda $v$, obtaining $v' = \{x, y, z\}$, and thus benefiting from the fact that $\text{V}(\{x, y, z\}) = x$.

On the other hand, if $z\text{P}_i x$, then we can reverse what we did in the previous paragraph. That is, we take $v$ to be the agenda $\{x, y, z\}$, and note that voter $i$'s preference for $z$ over $x$ is thwarted by the fact that $\text{V}(\{x, y, z\}) = x$. But now, if voter $i$ has agenda-setting power, he or she can delete $y$ from the agenda $v$, obtaining $v' = \{x, z\}$, and thus benefiting from the fact that $\text{V}(\{x, z\}) = z$. Because ballots are linear, this completes the proof of Claim 1.

Claim 2. R is complete.

Proof. If $x\text{R}y$ fails then $\text{V}(\{x, y\}) \neq x$. Because V is resolute, this means $\text{V}(\{x, y\}) = y$, and so $y\text{R}x$, as desired.

Claim 3. For any agenda $v$,

$$\text{V}(v) = x \quad \text{iff } x \in v \quad \text{and} \quad \forall y \in v\ [y \neq x \Rightarrow x\text{R}y].$$

Proof. Assume first that $\text{V}(v) = x$ and that $y \in v$ with $y \neq x$. If $x\text{R}y$ fails, then $\text{V}(\{x, y\}) = y$. Fix a voter $i$, and suppose first that $x\text{P}_i y$. Then, if $v = \{x, y\}$, voter $i$'s preference for $x$ over $y$ is initially thwarted by the fact that $\text{V}(\{x, y\}) = y$. But if voter $i$ has agenda-setting power, he or she can expand the agenda to $v$, and thus benefit from the fact that $\text{V}(v) = x$. Similarly if $y\text{P}_i x$, then, voter $i$'s preference for $y$ over $x$ is thwarted by the fact that $\text{V}(v) = x$. But if voter $i$ has agenda-setting power, he or she can shrink the agenda to $\{x, y\}$, and thus benefit from the fact that $\text{V}(\{x, y\}) = y$.

[13] Let's see why we lose no generality. Suppose that we had $\text{V}(\{x, y, z\}) = y$ instead. Then we could replace the first sentence in the proof of the claim with the following equivalent version: "Suppose that $y\text{R}z$ and $z\text{R}x$, and assume, for contradiction, that $y\text{R}x$ fails." We would then have $y$ playing the same role that $x$ plays in the proof above. A similar remark applies if $\text{V}(\{x, y, z\}) = z$.

Conversely, assume that $x \in v$ and $\forall y \in v\ [y \neq x \Rightarrow x\mathrm{R}y]$. Assume that $\mathrm{V}(v) = z$ for some $z \neq x$. But then, by the previous paragraph, we'd have that $\forall y \in v\ [y \neq z \Rightarrow z\mathrm{R}y]$. In particular, we'd have that $x\mathrm{R}z$ and $z\mathrm{R}x$, so $\mathrm{V}(\{x, z\}) = x$ and $\mathrm{V}(\{x, z\}) = z$; a clear contradiction. This completes the proof of Theorem 2.4.2.[14] □

We could easily restate Definition 2.4.1 so that it speaks of a social choice functon V being agenda-manipulable *by a voter* i. One could then say that a social choice fucntion is *weakly agenda-manipulable* if it is agenda-manipulable by at least one voter $i$, and *strongly agenda-manipulable* if it is agenda-manipulable by every voter $i$.

If we did this, what we are calling "agenda-manipulable" would then correspond to "weakly agenda-manipulable" in this sense. But it is easy to see that the proof of Theorem 2.4.2 goes through with either notion, and so the two are equivalent – something that is quite trivial to see in its own right.

This completes our discussion of agenda manipulability. Throughout the remainder of the book, we consider only the kind of manipulation in which a single voter achieves a preferred election outcome by submitting a disingenuous ballot. We begin in Chapter 3 with the easiest context: Resolute social choice functions.

## 2.5 Exercises

(1) [S] Suppose that **P** is a linear (A, $n$)-profile giving the true preferences of voter $i$. Let P be the relation defined on sets of alternatives by

$$\mathrm{XPY} \text{ iff } \max_i(\mathrm{X}, \mathbf{P})\, \mathrm{P}_i \min_i(\mathrm{Y}, \mathbf{P}).$$

Prove that R is neither irreflexive nor transitive.

(2) [S] Suppose that **P** is a linear (A, $n$)-profile giving the true preferences of voter $i$, and that X and Y are sets of alternatives. Prove that voter $i$ prefers X to Y in the sense of weak-dominance manipulation iff one of the following three conditions holds:

(i) $\mathrm{X} = \{x\}$ and $\mathrm{Y} = \{y\}$ and $x\mathrm{P}_i y$.
(ii) $\max_i(\mathrm{X}, \mathbf{P})\, \mathrm{P}_i \min_i(\mathrm{X}, \mathbf{P})\, \mathrm{R}_i \max_i(\mathrm{Y}, \mathbf{P})$.
(iii) $\min_i(\mathrm{X}, \mathbf{P})\, \mathrm{R}_i \max_i(\mathrm{Y}, \mathbf{P})\, \mathrm{P}_i \min_i(\mathrm{Y}, \mathbf{P})$.

[14] The relation R defined in the proof of Theorem 2.4.2 is known in the literature as the "base relation." For a great deal of related material, the reader can start with Sen's 1971 paper (available in Sen, 1982) "Choice Functions and Revealed Preferences."

(3) [S] Prove that single-winner manipulability implies weak dominance manipulability, that weak dominance manipulability implies manipulability by optimists or pessimists, and that manipulability by optimists or pessimists implies expected-utility manipulation. Hint: For the first two implications, use Exercise 2.

(4) [S] Suppose that **P** is a profile giving the true preferences of voter $i$, and that X and Y are sets of alternatives. For every $z \in A$, let $G_i(z) = \{x \in A: xR_iz\}$. Prove that X is preferred to Y in the sense of expected-utility if there exists some $z \in A$ such that

$$|X \cap G_i(z)|/|X| > |Y \cap G_i(z)|/|Y|.$$

(It turns out that the converse is also true, and follows from results later in the book.)

(5) Prove that if $aP_ibP_ic$, and $X = \{a, c\}$ and $Y = \{a, b, c\}$, then X can be preferred to Y in terms of expected utility and Y can be preferred to X in terms of expected utility. (Hint: Use Exercise 4.)

(6) Suppose $P_i$ is a linear ballot in which $a_1P_ia_2P_i \dots P_ia_k$. Let $X = \{a_2\}$ and let $Y = \{a_1, a_2, \dots, a_k\}$. Prove that there exists a utility function $u$ representing $P_i$ such that $E_{u,i}(X) > E_{u,i}(Y)$. Do the same for $X = \{a_2\}$ and $Y = \{a_1, \dots, a_k\}$.

(7) [S] Suppose $P_i$ is a linear ballot, X, $Y \subseteq A$ and $u$: $A \to \Re$ is a utility function representing $P_i$ such that $E_{u,i}(X) > E_{u,i}(Y)$. Let $r$ be an arbitrary real number and define $u'$: $A \to \Re$ by $u'(x) = u(x) + r$. Prove that $u'$ also represents $P_i$ and $E_{u',i}(X) > E_{u',i}(Y)$.

(8) [C/S] Let $A = \{a, b, c\}$ and $n = 3$. Let V be the voting rule for (A, $n$) wherein the winner is the alternative with the greatest total of first and second-place votes, with ties broken, where possible, by the number of first-place votes. (This is actually an example of what Young (1975) calls a "tie-breaking scoring system.")
   (a) Show that V is not resolute, but that it is single-winner manipulable.
   (b) Show that V satisfies Pareto (and note that it also satisfies anonymity, neutrality, and monotonicity).

(9) [C/S] Let $A = \{a, b, c\}$ and $n = 3$. Let V be the voting rule for (A, $n$) wherein the winner is the alternative with the greatest number of first- and second-place votes. Prove that V is weak-dominance manipulable, but not single-winner manipulable.

(10) [S] Show that if $|A| = 3$, then for every $n$, the Borda count for (A, $n$) is not single-winner manipulable.

(11) [C] Show that if $|A| \geq 4$, then for every $n$, the Borda count for (A, $n$) is single-winner manipulable.

(12) [C/S] Show that if $|A| \geq 3$, then the plurality rule for $(A, n)$ is weak-dominance manipulable iff $n \geq 4$.

(13) [C] Show that If $|A| = 3$ and $n \geq 3$, then the plurality rule for $(A, n)$ is min-weak dominance manipulable iff $n \neq 3, 4$ or 6.

(14) [C] Show that if $|A| \geq 3$ and $n \neq 2$ or 4, then the plurality rule for $(A, n)$ is manipulable by pessimists. (Hint: Build on Exercise 10.)

(15) [C] Show that if $|A| \geq 3$ and $n \geq 3$, the Condorcet rule is manipulable by both optimists and pessimists. (Hint: Find suitable profiles for $n = 3$ and $n = 4$, and then show that if such profiles exist for $n$, then they also exist for $n + 2$.)

(16) [S] Show that the Condorcet rule is never single-winner manipulable.

(17) [C/S] Show that if $|A| \geq 3$ and $n \geq 3$, the nomination-with-second rule is manipulable by pessimists iff $|A| \geq n$, and manipulable by optimists iff $n \geq 4$.

(18) [C] Show that if $|A| \geq 3$ and $n \geq 3$, the near-unanimity rule is manipulable by pessimists.

(19) [S] Prove that if a voter prefers $a$ to $b$ to $c$, then there exists a utility function realizing these preferences such that his or her expected utility from $\{a, b, c\}$ is higher than from $\{a, c\}$ and another utility function realizing these preferences such that his or her expected utility from $\{a, c\}$ is higher than from $\{a, b, c\}$.

(20) [C] Show that if $|A| \geq 3$ and $n \geq 3$, the Pareto for $(A, n)$ is expected-utility manipulable. (Hint: Use Exercise 16.)

(21) [S] Show that if $|A| = 3$ and $n \geq 3$, the weak Condorcet rule is not single-winner manipulable.

(22) [C/S] Show that if $|A| \geq 4$ and $n \geq 3$, the weak Condorcet rule is single-winner manipulable iff $n$ is even.

(23) [C] In the proof of Theorem 2.2.1, prove that the election winners are what they are advertised to be.

(24) [S] Prove that with the Condorcet rule, one cannot simultaneously improve the min and the max of the set of winners.

(25) [C] In the proof of Theorem 2.2.2, prove that the election winners are what they are advertised to be.

(26 [C] Prove that if $|A| = 3$ and $n = 5$, the iterated plurality rule is single-winner manipulable. (Hint: Suitable profiles occur in the proof of Theorem 2.2.2.)

(27) [C] Find the eighth single-winner manipulable voting rule not covered by Theorems 2.2.1 and 2.2.2.

(28) [S] A voter is a "dummy" for a voting rule if his or her ballot has no effect on the outcome of the election. For example, a dictatorship can be

thought of as arising from the unanimity rule with one voter by the addition of dummies. Prove that manipulability is unaffected by the addition or subtraction of dummies.

(29) [C/S] Show that the unanimity rule for $n \geq 2$ can't be manipulated by an optimist. Do the same for an oligarchy O, assuming that $|O| \geq 2$. Exercise 28 is relevant.

(30) [S] Prove that the omninomination rule and a triumvirate can't be manipulated by optimists or pessimists. Exercise 28 is relevant.

(31) [C] Prove that the omninomination rule and a triumvirate are expected-utility manipulable if $|A| \geq 3$ and $n \geq 3$.

(32) [S] Prove that a duumvirate is not expected-utility manipulable.

(33) [S] Reconcile the fact that the plurality rule is not single-winner manipulable with the first paragraph of the preface.

(34) [T] Use the Gibbard–Satterthwaite theorem to show that, for every $n \geq 2$, the Condorcet rule for $(A, n)$ is not resolute if $|A| \geq 3$. (Compare this with one of the exercises in Chapter 1.)

(35) [S/T] Prove that if $|A| = 2$ and $n$ is arbitrary, then a resolute voting rule for $(A, n)$ is manipulable iff it is monotone.

(36) [T] Prove that for two alternatives and an odd number of voters, majority rule is the only voting rule that is resolute, anonymous, neutral, and non-manipulable. (An exercise in Chapter 1 is relevant.)