zero by choosing a sufficiently large sample. In the next section we move from restrictive hypothetical distributions that predict a prevalence of majority cycles to the other extreme, namely restrictive theoretical domain constraints that rule out the occurrence of majority cycles. We then replace domain restrictions by restrictions on the distribution of preferences which we expect to be behaviorally descriptive of actual data.

1.2 NET VALUE RESTRICTION AND NET PREFERENCE MAJORITY

Now we introduce basic concepts such as weak stochastic transitivity and the weak utility model, as well as redefine concepts from the social choice literature such as "Condorcet winner" and "majority preference relations" in terms of probabilistic representations of linear order preferences. Then we state Sen's (1969, 1970) value restriction condition (Definition 1.2.7) and provide a probabilistic reformulation (Definition 1.2.8). We also generalize Sen's value restriction condition to what we call "net preference probabilities" (Definitions 1.2.10 and 1.2.13), so as to provide necessary and sufficient conditions for transitive majority preferences based on individual linear order preferences (Theorem 1.2.15).

1.2.1 Majority Rule and Probabilistic Preferences

A majority vote is *transitive* if the following property holds: Whenever candidate c has a majority over d and d has a majority over e, then c has a majority over e. Unless we explicitly ask voters to perform paired comparisons, it is not quite clear what this statement should mean in general. Yet, hardly any empirical ballots or survey responses provide full information on all paired comparisons. Thus, we need to take a more general perspective, as we do now.

Transitivity of votes is closely related to what the psychological and statistical choice literature calls "weak stochastic transitivity," and to the well-known "weak utility model" (Luce and Suppes, 1965). This latter model assumes that in a binary choice paradigm each paired comparison has a well-defined probability of a choice for each alternative (i.e., the choice of a given alternative is the outcome of a Bernoulli trial). The following definitions are from Luce and Suppes (1965).

Suppose that an individual (possibly drawn at random from a population) is asked to choose one candidate from a pair of candidates. Let p_{cd} denote the probability of a choice of c when c and d are being offered.

Definition 1.2.1 A *weak utility model* is a set of binary choice probabilities for which there exists a real-valued function w over C such that

$$p_{cd} \geq \frac{1}{2} \Leftrightarrow w(c) \geq w(d).$$

When C is finite, then the weak utility model is equivalent to weak stochastic transitivity of the binary choice probabilities, which we define next.

Definition 1.2.2 *Weak stochastic transitivity of binary choice probabilities* holds when

$$p_{cd} \geq \frac{1}{2} \quad \& \quad p_{de} \geq \frac{1}{2} \quad \Longrightarrow \quad p_{ce} \geq \frac{1}{2}.$$

Throughout this section we assume that individual preferences take the form of linear orders. This assumption will be dropped in Chapter 2. We write Π for the collection of all (strict) linear orders over C. For a given probability distribution $\pi \mapsto \mathbb{P}(\pi)$ over Π, we write $\mathbb{P}_{cd} = \sum_{(c,d)\in\pi} \mathbb{P}(\pi)$ for the marginal pairwise ranking probability of c being ranked ahead of d.

There exists a substantial literature trying to explain probabilities of (observable) binary choices by probabilities of (latent and unobserved) rankings through

$$p_{cd} = \mathbb{P}_{cd}. \tag{1.2}$$

Given a set of binary choice probabilities, it is not trivial to answer the question whether probabilities on rankings exist that satisfy (1.2). This question is commonly studied under the label "binary choice problem" and plays an important role in mathematical psychology as well as in operations research (Block and Marschak, 1960; Bolotashvili et al., 1999; Campello de Souza, 1983; Cohen and Falmagne, 1990; Dridi, 1980; Fishburn, 1990, 1992; Fishburn and Falmagne, 1989; Gilboa, 1990; Grötschel et al., 1985; Heyer and Niederée, 1989, 1992; Koppen, 1995; Marley, 1990; Marschak, 1960; McFadden and Richter, 1970; Suck, 1992).

In the probabilistic framework it is appealing and straightforward to define an aggregate preference relation through "c is aggregately preferred to d if and only if the choice probability $p_{cd} \geq \frac{1}{2}$ (in the Bernoulli trial)." Such a preference relation is transitive if and only if weak stochastic transitivity holds. Thus, for probabilistic binary choice (with $|C|$ finite), the existence of a transitive social welfare order, weak stochastic transitivity, and the weak utility model are equivalent. The function w in Definition 1.2.1 thus defines what we call a (majority) *social welfare function.*

Another important normative concept in the social choice literature, besides that of a majority weak order guaranteed by weak stochastic transitivity, is that of the "majority winner." A *Condorcet winner* (also known as *Condorcet candidate*, or a *majority winner*) is usually defined as the candidate(s) (if they exist) who would receive majority support against every other candidate if they were to compete pairwise (Black, 1958; Condorcet, 1785; Felsenthal et al., 1990; Young, 1986, 1988). The Condorcet winner is the most commonly accepted normative criterion for a social choice procedure that is required to select a single alternative.[17]

Since we usually lack data on binary comparisons, we formally define weak stochastic transitivity and the concept of a Condorcet winner in terms of (latent and unobserved) probabilistic rankings.[18]

Definition 1.2.3 A probability distribution $\mathbb{P}$ on Π satisfies *weak stochastic transitivity (for rankings)* if and only if the induced marginal (pairwise) ranking probabilities satisfy

$$\mathbb{P}_{cd} \geq \frac{1}{2} \quad \& \quad \mathbb{P}_{de} \geq \frac{1}{2} \quad \Longrightarrow \quad \mathbb{P}_{ce} \geq \frac{1}{2}.$$

Definition 1.2.4 Given a probability $\mathbb{P}$ on Π, candidate $c \in C$ is a *Condorcet winner* if and only if

$$\mathbb{P}_{cd} \geq \frac{1}{2} \qquad \forall d \in C - \{c\}.$$

These concepts of weak stochastic transitivity and of a Condorcet candidate are compatible with the idea that, if the voters were indeed asked to do a paired comparison, they would actually base their decision on a latent preference ranking and would choose the alternative (in the Bernoulli process) that is ranked ahead of the other in the sampled preference ranking.[19]

[17] The arguably second most commonly accepted normative benchmark, and main competitor of the Condorcet winner, is the Borda winner. The latter is strongly advocated by some researchers, e.g., Saari (1994; 1995).

[18] In fact, we usually also lack information on full rankings. The subsequent chapters discuss this situation in detail.

[19] It should be noted that the reverse approach has also been modeled where only paired comparison probabilities are given, and ranking probabilities are constructed from those paired comparison probabilities (e.g., Marley, 1968).

Definition 1.2.5 Consider a probability $\mathbb{P}$ on Π. We define a *weak majority preference relation* $\succsim$ and a *strict majority preference relation* $\succ$ through

$$c \succsim d \quad \Leftrightarrow \quad \mathbb{P}_{cd} \geq \mathbb{P}_{dc} \Leftrightarrow \mathbb{P}_{cd} \geq \frac{1}{2}, \tag{1.3}$$

$$c \succ d \quad \Leftrightarrow \quad \mathbb{P}_{cd} > \mathbb{P}_{dc} \Leftrightarrow \mathbb{P}_{cd} > \frac{1}{2}. \tag{1.4}$$

Observation 1.2.6 *The weak majority preference relation $\succsim$, as defined in Definition 1.2.5, is reflexive and strongly complete (and thus complete). Therefore, $\succsim$ is a weak order if and only if it is transitive. The strict majority preference relation $\succ$, as defined in Definition 1.2.5, is asymmetric (and thus antisymmetric). Therefore, $\succ$ is a strict weak order if and only if it is negatively transitive. More generally, $\succ$ is a strict partial order if and only if it is transitive. Furthermore, $\succ$ is a strict weak order if and only if $\succsim$ is a weak order.*

The proof is in Appendix C.

When $\succsim$ is transitive, then $\succ$ is also transitive. However, $\succ$ may be transitive when $\succsim$ is not: Suppose that preferences are linear orders and that $\mathbb{P}(\{(a,b),(b,c),(a,c)\}) = \mathbb{P}(\{(a,b),(c,b),(c,a)\}) = \frac{1}{2}$. Then, $\succ = \{(a,b)\}$, which is transitive. On the other hand, $\succsim = \{(a,a), (a,b),(a,c),(b,b),(b,c),(c,c),(c,a),(c,b)\}$, which is intransitive.

Chapter 2 develops more general definitions of Condorcet winner and majority preference relations for a broad range of deterministic and/or probabilistic representations of preference or utility.

Sen's "value restriction" is a sufficient condition for the existence of a transitive social welfare ordering. We first state its formal definition, and then translate the condition into probabilistic terms.

Definition 1.2.7 Suppose that each voter has a strict linear preference order. Given a collection of voters, a triple of alternatives satisfies NW (*never worst*) if and only if there is one alternative among the three that is not ranked worst (i.e., third) among the three by any of the voters; NM (*never middle*) if and only if there is one alternative among the three that is not ranked in the middle (i.e., second) among the three by any of the voters; NB (*never best*) if and only if there is one alternative among the three that is not ranked best (i.e., first) among the three by any of the voters. Sen's *value restriction* holds if and only if every triple of alternatives satisfies either NW, NM, or NB.

The underlying intuition of value restriction is that in each triple of candidates there should be at least one about which all voters agree that s/he is not the worst, or not the middle, or not the best.

We now translate the *NB*, *NM*, and *NW* conditions into probabilistic terms for linear order preferences. (In Chapter 2, we further generalize these concepts to a much broader framework.)

1.2.2 Probabilistic Reformulation and Generalizations of Sen's Value Restriction

Definition 1.2.8 Consider a probability $\mathbb{P}$ on Π. For any given triple of alternatives, we say that the marginal ranking probabilities induced by $\mathbb{P}$ on that triple satisfy $NW(c)$ if and only if the (marginal) probability for c to be ranked worst (in the triple) is zero. When $NW(c)$ holds, c is said to be (*almost surely*, abbreviated a.s.) *never worst*. $NM(c)$ and $NB(c)$ are defined analogously. $\mathbb{P}$ is *(a.s.) value restricted* if and only if in each triple $\{x, y, z\} \subseteq C$ there exists c with either $NW(c)$, $NM(c)$, or $NB(c)$. In that case, we also say that *(a.s.) value restriction* holds.

The following theorem is a variation of Sen's theorem on value restriction (Sen, 1966, 1969, 1970), generalized to probabilistic terms.

Theorem 1.2.9 *Given a probability $\mathbb{P}$ on Π, consider the relations $\succsim$ and $\succ$ of Definition 1.2.5. If $\mathbb{P}$ is (a.s.) value restricted, then 1) the weak majority preference relation $\succsim$ is a weak order, 2) the strict majority preference relation $\succ$ is a strict weak order, and 3) if $\mathbb{P}_{cd} \neq \mathbb{P}_{dc}, \forall c \neq d$, then the strict majority preference relation $\succ$ is a strict linear order. Thus, (a.s.) value restriction implies transitivity.*

We now move from domain restrictions to restrictions on the distribution of preferences, or, more specifically, on the "net preference probabilities" over the full domain. The following definitions are critical throughout the rest of this section. (Again, Chapter 2 generalizes these concepts much further.)

Definition 1.2.10 Given a probability $\mathbb{P}$ on Π, and denoting by π^{-1} the reverse order of π, the *net ranking probability* (*net preference probability*) $N^{\mathbb{P}}$ (induced by $\mathbb{P}$) is defined as

$$N^{\mathbb{P}}(\pi) = \mathbb{P}(\pi) - \mathbb{P}(\pi^{-1}).$$

The *net margins* (*net pairwise preference probabilities*) are defined as $N\mathbb{P}_{cd} = \mathbb{P}_{cd} - \mathbb{P}_{dc}$. (See Feld and Grofman, 1988, for a similar concept.)

Net marginal ranking probabilities of triples are defined analogously. We also write $N\mathbb{P}_{\pi}$ for $N\mathbb{P}(\pi)$, and $N\mathbb{P}_{cde}$ for the net marginal ranking probability that c is ranked before both d and e, and that d is ranked before e. The following observation follows immediately from these notational conventions.

Observation 1.2.11 *Given $N\mathbb{P}$ on Π and $\succsim$ and $\succ$ on C as above, we have*

$$c \succsim d \Leftrightarrow N\mathbb{P}_{cd} \geq 0, \quad c \succ d \Leftrightarrow N\mathbb{P}_{cd} > 0.$$

We now define NW, NM, and NB for net probabilities on linear orders.

Definition 1.2.12 Given $N\mathbb{P}$ on Π as before, for any triple $\{c, d, e\} \subseteq C$,

$$\begin{aligned} N\mathbb{P}\,\text{satisfies}\,NW(c) &\Leftrightarrow N\mathbb{P}_{edc} \leq 0 \;\&\; N\mathbb{P}_{dec} \leq 0, \\ N\mathbb{P}\,\text{satisfies}\,NM(c) &\Leftrightarrow N\mathbb{P}_{ecd} \leq 0 \;\&\; N\mathbb{P}_{dce} \leq 0 \Leftrightarrow N\mathbb{P}_{ecd} = 0, \\ N\mathbb{P}\,\text{satisfies}\,NB(c) &\Leftrightarrow N\mathbb{P}_{cde} \leq 0 \;\&\; N\mathbb{P}_{ced} \leq 0. \end{aligned}$$

When $N\mathbb{P}$ satisfies $NB(a)$, we often also say for short that $NetNB(a)$ is satisfied. The same applies to the other conditions and choice alternatives.

Definition 1.2.13 $N\mathbb{P}$ is *marginally value restricted* for the triple $\{x, y, z\} \subseteq C$ if and only if there exists an element $c \in \{x, y, z\}$ such that $N\mathbb{P}$ satisfies $NW(c)$ or $NB(c)$ or $NM(c)$. If this property is satisfied, then *marginal net value restriction* holds on the triple $\{x, y, z\}$. *Net value restriction* holds on C if marginal net value restriction holds on each triple. When net value restriction holds, we also say that the *net value restriction condition* is satisfied.

Remark. If $N\mathbb{P}$ on Π satisfies $NW(c)$ for a triple $\{c, d, e\} \subseteq C$, then

$$\begin{aligned} N\mathbb{P}_{ecd} \leq 0 &\Rightarrow N\mathbb{P}\,\text{satisfies}\; NB(e), \\ N\mathbb{P}_{ecd} \geq 0 &\Rightarrow N\mathbb{P}\,\text{satisfies}\; NB(d). \end{aligned}$$

Similarly, $NB(c)$ implies either $NW(d)$ or $NW(e)$. Also, $NM(c)$ means that $N\mathbb{P}_{ecd} = 0$ and thus it means that at most two rankings have strictly positive $N\mathbb{P}$ values, and that $NW(d)$ or $NW(e)$ holds. At most three elements in

$\{cde, ced, dce, dec, ecd, edc\}$ have strictly positive net preference probabilities. Furthermore, net value restriction is weaker than value restriction:

- $\mathbb{P}$ satisfies $NW(c) \Rightarrow \mathcal{NP}$ satisfies $NW(c)$, but not conversely,
- $\mathbb{P}$ satisfies $NB(c) \Rightarrow \mathcal{NP}$ satisfies $NB(c)$, but not conversely,
- $\mathbb{P}$ satisfies $NM(c) \Rightarrow \mathcal{NP}$ satisfies $NM(c)$, but not conversely.

Clearly, domain restrictions imply distributional restrictions, but the converse does not generally hold.

We need a further definition before we can state our key theorem on necessary and sufficient conditions for weak stochastic transitivity on probability distributions over linear orders.

Definition 1.2.14 Given $\mathcal{NP}$ on Π as before, $\pi \in \Pi$ has a *net preference majority* if and only if

$$\mathcal{NP}(\pi) > \sum_{\substack{\pi' \in \Pi - \{\pi\}, \\ \mathcal{NP}(\pi') > 0}} \mathcal{NP}(\pi'). \tag{1.5}$$

Similarly, for any triple $\{c, d, e\} \subseteq C$, cde has a *marginal net preference majority* if and only if

$$\mathcal{NP}_{cde} > \sum_{\substack{\pi' \in \{ced, dce, dec, ecd, edc\}, \\ \mathcal{NP}_{\pi'} > 0}} \mathcal{NP}_{\pi'}.$$

We say that the *net majority condition* holds if there is a linear order that has a net preference majority.

The following theorem is similar in spirit to Lemma 2 of Feld and Grofman (1986b).[20]

Theorem 1.2.15 *The weak majority preference relation $\succsim$ defined in Definition 1.2.5 is transitive if and only if for each triple $\{c, d, e\} \subseteq C$ at least one of the following two conditions holds:*

1. *$\mathcal{NP}$ is marginally value restricted on $\{c, d, e\}$ and, in addition, if at least one net preference is nonzero then the following implication is true (with possible relabelings):*

$$\mathcal{NP}_{cde} = 0 \Rightarrow \mathcal{NP}_{dce} \neq \mathcal{NP}_{ced}.$$

[20] Note that their treatment omits certain knife-edge situations caused by possible ties.

2. *$\exists\pi_0 \in \{cde, ced, dce, dec, ecd, edc\}$ such that π_0 has a marginal net preference majority.*

Similarly, the strict majority preference relation $\succ$ is transitive if and only if for each triple $\{c, d, e\} \subseteq C$ at least one of the following two conditions holds:

1. *NP is marginally value restricted on $\{c, d, e\}$.*
2. *$\exists\pi_0 \in \{cde, ced, dce, dec, ecd, edc\}$ such that π_0 has a marginal net preference majority.*

The proof is in Appendix C.

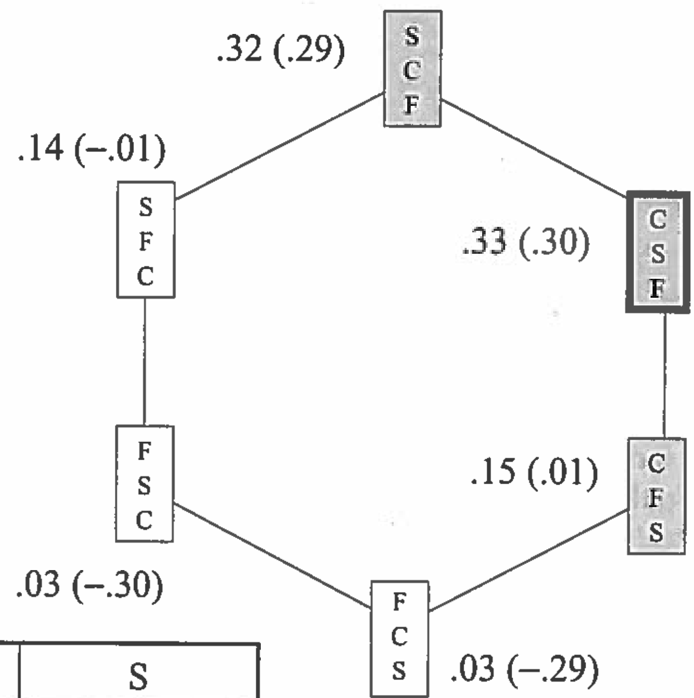
1.3 EMPIRICAL ILLUSTRATIONS

This section provides brief empirical illustrations based on survey preference data. First, these data can be shown clearly not to originate from an impartial culture. Also, for these data, Sen's value restriction is violated, but nevertheless, majority preferences are transitive. We show how our net value restriction condition (and, incidentally, not the net majority condition) accounts for the absence of majority cycles in these data.

It is difficult to find empirical data that provide either complete paired comparisons or complete linear orders of all choice alternatives as technically required by any standard definition of majority rule, including the one we use in this section. We consider three national survey data sets from Germany in which complete linear orders of three major parties were reported for all respondents (Norpoth, 1979). These three major parties are the Social Democratic party (S), the Christian Democratic parties (C), and the Free Democratic party (F). The data under consideration are 1969, 1972, and 1976 German National Election Study (GNES) data sets.

Figure 1.1 displays the results for the 1969 GNES (this survey distribution was reported by Norpoth, 1979). The graph shows all possible linear order preference states, as well as the relative frequencies of their occurrences in the 1969 GNES. We identify relative frequencies with probabilities and provide the net probabilities in parentheses. The inset table provides the pairwise net probabilities. For instance, $NP_{CF} = .6$. Linear orders and paired comparisons with positive net probabilities are shaded in grey.

First of all, we can test the hypothesis that the survey data set originated from a uniform distribution over linear orders. We use the following likelihood ratio test. Writing N_π for the frequency with which the linear order π was observed in the survey, and writing $N = \sum_{\pi\in\Pi} N_\pi$ for the



NP	F	S
C	.6	.02
F		–.58

FIGURE 1.1: Net value restriction in the 1969 GNES. The graph provides all ranking probabilities and net probabilities (in parentheses). The inset table provides the pairwise net probabilities NP_{CF}, NP_{CS}, and NP_{FS}. Linear orders and paired comparisons with positive net probabilities are shaded in grey. The linear order with boldface frame is $\succ$, the majority preference relation.

total sample size (in the 1969 survey, $N = 818$), the likelihood L_{IC} of the observed data under the impartial culture assumption is given by

$$L_{IC} = \left(\frac{1}{6}\right)^N,$$

whereas the likelihood L_{MN} of the data under an unconstrained multinomial distribution $\pi \mapsto p_\pi$ is given by

$$L_{MN} = \prod_{\pi \in \Pi} p_\pi^{N_\pi}. \tag{1.6}$$

The maximum likelihood estimates $\hat{p}_\pi$ of the values p_π are simply the relative frequencies in the observed data, i.e., $\hat{p}_\pi = \frac{N_\pi}{N}$. Therefore, to test the hypothesis that the observed data are a random sample from a uniform distribution, as opposed to a random sample from an unconstrained multinomial distribution, we simply substitute the maximum likelihood estimates $\hat{p}_\pi$ for the probabilities p_π in Equation (1.6) and compute the we get $\frac{\partial F}{\partial \sigma_2} < 0$ and so $\sigma_2 = 1$ at the minimum. Thus we have the following necessary condition for F to reach its minimum:

$$\sigma_1 = \sigma_2 = \sigma_3 = 1.$$

Now the problem is reduced to finding the minimum of

$$\arcsin(\omega_1) + \arcsin(\omega_2) + \arcsin(\omega_3)$$

under the constraints

$$\omega_1 + \omega_2 + \omega_3 = 1, \quad |\omega_i| \leq 1.$$

Substituting $\omega_1 = 1 - \omega_2 - \omega_3$, we thus need to find the minimum of

$$\arcsin(1 - \omega_2 - \omega_3) + \arcsin(\omega_2) + \arcsin(\omega_3).$$

Now

$$\frac{\partial F}{\partial \omega_3} < 0 \Leftrightarrow 1 - \omega_2 - \omega_3 > \omega_3,$$

$$\frac{\partial F}{\partial \omega_3} > 0 \Leftrightarrow \omega_3 > 1 - \omega_2 - \omega_3.$$

Since the minimum has to be at the point where the derivative changes its sign from negative to positive, we conclude that at the minimum

$$\omega_3 = \frac{1 - \omega_2}{2}.$$

However, from the fact that $|\omega_i| \leq 1$, and that we are dealing with the case $\omega_2 > 0$, we obtain that $\omega_2 < 1$, and so we have $\omega_i > 0, i = 1, 2, 3$. Since the function $\arcsin(x)$ is increasing and convex in x for $x \geq 0$, F reaches its minimum for $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$. Thus, at the minimum, $\sigma_i = 1$, $\omega_i = \frac{1}{3}$, $i = 1, 2, 3$. The desired result then follows by substituting these values in (C.1) and (C.2), and by using the definition of p_B. Thus, we obtain that the probability of cycles reaches its maximum for

$$p_{abc} = p_{acb} = p_{cab} = p_{bac} = p_{bca} = p_{cba} = \frac{1 - p_{aEbEc}}{6},$$

$$0 \leq p_{aEbEc} < 1. \quad \blacksquare$$

Proof of Observation 1.2.6

Reflexivity of $\succsim$: $c \succsim c \Leftrightarrow NP_{cc} \geq 0$. The latter holds since $NP_{cc} = 0, \forall c \in C$.

Strong Completeness of $\succsim$: $\forall c, d \in C$, either $NP_{cd} \geq 0$ or $NP_{dc} = -NP_{cd} \geq 0$.

Irreflexivity of $\succ$: This follows immediately from the fact that $NP_{cc} = 0, \forall c \in C$.

Asymmetry of $\succ$: $\forall c, d \in C$, if $NP_{cd} > 0$ then $NP_{dc} = -NP_{cd} < 0$.

It is clear that $\succ$ is the asymmetric part of $\succsim$. The rest of the Observation follows immediately. ■

Proof of Theorem 1.2.15. Transitivity holds on C if and only if transitivity holds on each triple $\{c, d, e\}$ in C. Thus, there is no loss of generality in setting $C = \{c, d, e\}$ and $\Pi = \{cde, ced, dce, dec, ecd, edc\}$. Recall that at most three rankings have (strictly) positive net preference probabilities.

First, suppose that none is positive, i.e., that all net ranking probabilities are zero. Then transitivity holds because all alternatives are tied, i.e., $\succ = \emptyset$, $\succsim = C \times C$, and net value restriction holds (but there is no ranking with a net preference majority).

Second, suppose that exactly one net ranking probability NP_π is positive (i.e., four net ranking probabilities are zero). Then transitivity holds since $\succsim = \succ = \pi$. Net value restriction holds, with $NP_{cde} = 0 \Rightarrow NP_{dce} \neq NP_{ced}$ (including possible relabelings), and π has a net preference majority.

Third, suppose that exactly two net ranking probabilities are null, without loss of generality assume that $NP_{cde} = -NP_{edc} = 0$. Then $NM(d)$ holds, and therefore net value restriction also holds.

a) If $NP_{dce} > 0$ & $NP_{dec} > 0$ (and thus $NP_{dce} \neq NP_{ced}$) then transitivity follows:

$$NP_{dce} > NP_{dec} \Rightarrow \succsim = \succ = dce \text{ with a net preference majority,}$$
$$NP_{dce} = NP_{dec} \Rightarrow \succ = \{(d, c), (d, e)\}; \succsim = \{(d, c), (d, e), (c, e), (e, c)\},$$
$$NP_{dce} < NP_{dec} \Rightarrow \succsim = \succ = dec \text{ with a net preference majority;}$$

b) If $NP_{dce} < 0$ & $NP_{dec} < 0$ (and thus $NP_{dce} \neq NP_{ced}$) then transitivity follows:

$$NP_{dce} > NP_{dec} \Rightarrow \succsim = \succ = ced \text{ with a net preference majority,}$$
$$NP_{dce} = NP_{dec} \Rightarrow \succ = \{(c, d), (e, d)\}; \succsim = \{(c, d), (e, d), (c, e), (e, c)\},$$
$$NP_{dce} < NP_{dec} \Rightarrow \succsim = \succ = ecd \text{ with a net preference majority;}$$

c) If $NP_{dce} > 0$ & $NP_{dec} < 0$ then

$$NP_{dce} > NP_{ced} \Rightarrow \succsim = \succ = dce \text{ with a net preference majority,}$$
$$NP_{dce} < NP_{ced} \Rightarrow \succsim = \succ = ced \text{ with a net preference majority,}$$
$$NP_{dce} = NP_{ced} \Rightarrow \begin{cases} \succ = \{(c, e)\}; \\ \succsim = \{(d, c), (c, d), (d, e), (e, d), (c, e)\}; \quad (\dagger) \end{cases}$$

where (†) is a violation of transitivity for R.

d) If $NP_{dce} < 0$ & $NP_{dec} > 0$ then

$$NP_{ecd} > NP_{dec} \Rightarrow \succsim = \succ = ecd \text{ with a net preference majority,}$$
$$NP_{ecd} < NP_{dec} \Rightarrow \succsim = \succ = dec \text{ with a net preference majority,}$$
$$NP_{ecd} = NP_{dec} \Rightarrow \begin{cases} \succ = \{(e, c)\}; \\ \succsim = \{(d, c), (c, d), (d, e), (e, d), (e, c)\}; \quad (\ddagger) \end{cases}$$

where (‡) is a violation of transitivity for R.

FOURTH, the only remaining possibility is that three net probabilities are positive (and the others are negative, i.e., $NP_{xyz} = 0$ cannot occur). There are eight such cases:

$$NP_{cde} > 0 \ \& \ NP_{dce} > 0 \ \& \ NP_{ced} > 0, \tag{C.9}$$
$$NP_{cde} > 0 \ \& \ NP_{dce} > 0 \ \& \ NP_{dec} > 0, \tag{C.10}$$
$$NP_{cde} > 0 \ \& \ NP_{ecd} > 0 \ \& \ NP_{ced} > 0, \tag{C.11}$$
$$NP_{cde} > 0 \ \& \ NP_{ecd} > 0 \ \& \ NP_{dec} > 0, \tag{C.12}$$
$$NP_{edc} > 0 \ \& \ NP_{dce} > 0 \ \& \ NP_{ced} > 0, \tag{C.13}$$
$$NP_{edc} > 0 \ \& \ NP_{dce} > 0 \ \& \ NP_{dec} > 0, \tag{C.14}$$
$$NP_{edc} > 0 \ \& \ NP_{ecd} > 0 \ \& \ NP_{ced} > 0, \tag{C.15}$$
$$NP_{edc} > 0 \ \& \ NP_{ecd} > 0 \ \& \ NP_{dec} > 0. \tag{C.16}$$

The cases (C.9)–(C.11) and (C.14)–(C.16) are all equivalent through relabeling of alternatives: Starting each time from (C.9), the relabeling $c \leftrightarrow d$ yields (C.10), $d \leftrightarrow e$ yields (C.11), $c \to d \to e \to c$ yields (C.14), $c \to e \to d \to c$ yields (C.15), and $c \leftrightarrow e$ yields (C.16). Similarly, (C.12) is equivalent to (C.13) through, for instance, the relabeling $c \leftrightarrow e$. We thus need to consider only (C.9) and (C.12).

If (C.9) holds, then net value restriction holds, $c \succ e$ and, furthermore,

$$NP_{cde} < NP_{ced} + NP_{dce} \Rightarrow \begin{cases} d \succsim c \Rightarrow d \succ e \\ e \succsim d \Rightarrow c \succ d, \end{cases}$$
$$NP_{cde} \geq NP_{ced} + NP_{dce} \Rightarrow \succsim = \succ = cde,$$

each of which implies transitivity for both $\succsim$ and $\succ$.

If (C.12) holds (and thus net value restriction is violated) we obtain transitivity if and only if one of the three rankings *cde*, *ecd*, and *dec* has a net preference majority: Suppose that each of the three has a net probability strictly smaller than the sum of the other two. Then $\succsim = \succ = \{(c,d),(d,e),(e,c)\}$, a violation of transitivity. Also, if one of the three has a net probability equal to the sum of the other two, say $NP_{cde} = NP_{ecd} + NP_{dec}$, then $\succ = \{(c,d),(d,e)\}$, $\succsim = \{(c,d),(d,e),(c,e),(e,c)\}$, which both violate transitivity. ■

Proof of Observation 2.3.6. We provide a counterexample for strict weak orders, which suffices also as a counterexample for strict partial orders and more general settings. Let $NP\begin{pmatrix} b \\ c \\ a \end{pmatrix} = NP\begin{pmatrix} c \\ b \\ a \end{pmatrix} = NP\begin{pmatrix} c \\ a \\ b \end{pmatrix} = NP\begin{pmatrix} & b & \\ a & & c \end{pmatrix} = NP\begin{pmatrix} b & & c \\ & a & \end{pmatrix} = NP\begin{pmatrix} & c & \\ a & & b \end{pmatrix} = -.2$. It is straightforward to check that $NW(a)$ holds but no other net value restriction condition is satisfied. ■

Proof of Theorem 2.3.8. The proof is by counterexample. Take $NP\begin{pmatrix} a \\ b \\ c \end{pmatrix} = .004$, $NP\begin{pmatrix} b \\ a \\ c \end{pmatrix} = .003$, $NP\begin{pmatrix} a \\ c \\ b \end{pmatrix} = .002$, $NP\begin{pmatrix} b & & c \\ & a & \end{pmatrix} = .002$, with the remaining net probabilities equal to zero. Then net value restriction does not hold, nor is there a binary relation with a net majority. Nevertheless, $\succ = \begin{matrix} a \\ b \\ c \end{matrix}$ and $\succsim = \succ \cup Id$ (where Id is the identity relation) are transitive social welfare orders. A graphical display of this counterexample is given in Figure 2.10. Notice that this result holds unaltered when we allow individuals to have arbitrary binary preferences: If a large proportion of the population has cyclic preferences, i.e., we add a net probability of the forward cycle to the above list, then the social welfare order remains unchanged in this example as long as $-.001 < NP\begin{pmatrix} & a & \\ & \bigcirc & \\ c & & b \end{pmatrix} < .007$. This can happen even when more than half of the population has cyclic preferences. ■

Proof of Theorem 2.3.9. i) This follows directly from Theorem 2.1.5 and Observation 2.3.5.
ii) The following is a counterexample: Suppose $P\begin{pmatrix}a\\b\end{pmatrix} = 1$. Then net value restriction holds, $\succ = \begin{matrix}a\\b\end{matrix}$, a semiorder, and thus transitive, but $\succsim = \{(a,a),(a,b),(a,c),(b,b),(b,c),(c,a),(c,b),(c,c)\}$, which is not transitive because $b \succsim c \succsim a$ but not $b \succsim a$. ■

Proof of Theorem 2.3.10. Suppose that $C = \{a,b,c\}$. To prove i) first notice that, up to a relabeling of the alternatives, there are only three possible strict weak orders with a net preference majority, namely $\begin{matrix}a\\b\\c\end{matrix}$, $\begin{matrix}a\ b\\c\end{matrix}$, or $\begin{matrix}a\\b\ c\end{matrix}$. Let $\begin{matrix}a\\b\\c\end{matrix}$ have a net majority. Then $\succ = \begin{matrix}a\\b\\c\end{matrix}$ is the social welfare order. If $\begin{matrix}a\ b\\c\end{matrix}$ has a net majority, then the social welfare order $\succ$ is transitive because it must be one of the following three strict weak orders: $\begin{matrix}a\\b\\c\end{matrix}$, $\begin{matrix}b\\a\\c\end{matrix}$, or $\begin{matrix}a\ b\\c\end{matrix}$. The proof for $\begin{matrix}a\\b\ c\end{matrix}$ follows the same logic.

To obtain a counterexample for ii), suppose that $P\begin{pmatrix}a\\b\end{pmatrix} = .7$, $P\begin{pmatrix}a\\b\\c\end{pmatrix} = P\begin{pmatrix}b\\c\\a\end{pmatrix} = P\begin{pmatrix}c\\a\\b\end{pmatrix} = .1$, with the remaining probabilities equal to zero. Then $\begin{matrix}a\\b\end{matrix}$ has a net preference majority. It is easy to check that $NP_{ab} = .8$, $NP_{bc} = .1$, $NP_{ca} = .1$, and thus we obtain majority cycles $a \succsim b \succsim c \succsim a$ and $a \succ b \succ c \succ a$ while neither $b \succsim a$ nor $b \succ a$ is the case, i.e., neither $\succsim$ nor $\succ$ is transitive. ■

Proof of Theorem 5.1.4. By (5.4), $P(\succ_S = \succ^*)$ has the following lower bound:

$$Err_N(N, \delta_1^*) - (M-1)Err_N(N, \delta_2^*) \leq P(\succ_S = \succ^*).$$