MANAGEMENT SCIENCE
Vol. 28, No. 2, February 1982
Printed in U.S.A.

Departmental Editor's Note

The paper, "Subjective Probability and the Theory of Games" by Professors Joseph B. Kadane and Patrick D. Larkey, and the "Comments" by Professor John C. Harsanyi, present opposing viewpoints on game theory. Similar views have been often discussed informally by Bayesian statisticians and game theorists, but we are glad to have an opportunity to present them more formally in *Management Science*. From the management scientist's perspective, the key issue concerns the usefulness of game theory in solving managerial decision problems. Correspondence on this topic is invited.

AMBAR G. RAO

SUBJECTIVE PROBABILITY AND THE THEORY OF GAMES*

JOSEPH B. KADANE† AND PATRICK D. LARKEY†

This paper explores some of the consequences of adopting a modern subjective view of probability for game theory. The consequences are substantial. The subjective view of probability clarifies the important distinction between normative and positive theorizing about behavior in games, a distinction that is often lost in the search for "solution concepts" which largely characterizes game theory since the work of von Neumann and Morgenstern. Many of the distinctions that appear important in conventional game theory (two-person versus n-person, zero-sum versus variable sum) appear unimportant in the subjective formulation. Other distinctions, such as single play versus repetitive-play games, appear to be more important in the subjective formulation than in the conventional formulation.
(GAME THEORY; RATIONALITY; BAYESIANISM; SUBJECTIVE PROBABILITIES)

> "Probability has often been visualized as a subjective concept more or less in the nature of an estimation. Since we propose to use it in constructing an individual, numerical estimation of utility, the above view of probability would not serve our purpose. The simplest procedure is, therefore, to insist upon the alternative, perfectly well founded interpretation of probability as frequency in the long run."
>
> von Neumann and Morgenstern [49, p. 19]

The Theory of Games and Economic Behavior (von Neumann and Morgenstern, [49]) has directly and indirectly spawned an enormous body of work. Theories of games are found in several disciplines including mathematics (Lucas, [26]), statistics (Blackwell and Girshick, [6]), economics (Schotter and Schwodiauer, [42], political science (Riker and Ordeshook, [36]) and social psychology (Miller and Steinfatt, [74]). Game theory has become a field in its own right with journals devoted primarily to the topic and academics pursuing careers as "game theorists."

*Accepted by Ambar G. Rao; received September 1980. This paper has been with the authors 2 months for 1 revision.
†Carnegie-Mellon University.

0025-1909/82/2802/0113$01.25

Normative and positive work on game theory has been motivated, in the main, by an interest in rigorously understanding human behavior in situations of strategic interaction, particularly conflict situations. The prospective applications of this understanding are in advising parties to conflict situations and in predicting the outcomes of conflicts. The work on game theory has been largely directed at constucting formal models and deducing, within given game structures (number of players, number of plays, form of pay-offs, etc.) and given assumptions about human behavior (usually a set of axioms or postulates for "rationality"), what the outcome of the games' play will be. These deduced outcomes are called "solution concepts."

In some of the disciplinary applications, games are explicitly constructed to serve as metaphors or analogies for naturally occurring conflict phenomena. In economics, game theoretic models have been used as metaphors for the market situations of duopoly, oligopoly, and perfect competition. Political scientists have used game theoretic frameworks to represent a variety of voting and other institutional decision contexts. In mathematics there is less concern with what the formal game theoretic models might represent empirically; game theory has proved to be a very rich source of intrinsically interesting mathematical problems. Some statistical decision theorists have adapted the minimax criterion from game theory to analyze statistical problems as if they were games against "nature."

There are other descendants of von Neumann and Morgenstern, although the lineage is less obvious. von Neumann and Morgenstern revived interest in cardinal utility theory and the principle of maximizing expected utility as a canon of rational behavior. However, their version of probability was "objective," probabilities derived from relative "frequencies in the long run." Thus, they proposed finding expected utility by integrating subjective utility with respect to objective probability. This line was explored and extended by Abraham Wald, particularly in *Statistical Decision Functions* [51], and further explored by Savage in his *Foundations of Statistics* [40]. Savage's work began as a defense of the von Neumann-Morgenstern-Wald minimax approach; he concluded by shifting to a subjective view of probability, upholding the principle of subjective utility integrated with respect to subjective probability. This work was the starting point for the modern Bayesian view of statistics and econometrics, as exemplified by the work reviewed in (Lindley, [25]), and the essays in (Fienberg and Zellner, eds., [14]) and (Zellner, ed., [54]).

It is a curiosity of intellectual history that these two lines of inquiry have had so little to do with one another despite their common heritage. There has been some slight contact, (Harsanyi, [15]), (Auman and Maschler, [3]), (Harsanyi and Selten, [18]), (Mertens and Zamir, [28]), (Zamir, [53]), (Ponssard and Zamir, [31]), (Ponssard, [32], (Kohlberg [22]), (Sorin, [46]), having to do with games in which some of the payoffs are unknown.

The primary purpose of this paper is to explore some of the consequences for game theory of taking the modern subjective view of probability in the sense of Savage and the writers who have followed in that tradition.[1] The paper is offered in the spirit of a query to game theorists: Why has game theory followed the course of development that it has? As individuals interested in the development of useful prescriptive and

[1] Morris H. DeGroot [10, p. 4] has summarized the modern view as follows:

> "According to the subjective, or personal, interpretation of probability, the probability that a person assigns to a possible outcome or some process represents his own judgment of the likelihood that the outcome will be obtained. This judgment will be based on that person's beliefs and information about the process. Another person, who may have different beliefs or different information, may assign a different probabiltiy to the same outcome."

predictive theories of individual and collective decision-making behavior, we have often been perplexed by the emphases in game theoretic research. In particular, we do not understand the search for solution concepts that do not depend on the beliefs of each player about the others' likely actions and yet are so compelling that they will become the obvious standard of play for all those who encounter them. The paper is exploratory. We sketch some of the more obvious implications of a subjective probability view for game theory.

Harsanyi [17], [15] was among the first to characterize game players as "Bayesians." For a variety of reasons including the perceived mathematical difficulties from "an infinite regress in reciprocal expectations on the part of players" and the perceived necessity of deriving determinate solution concepts, Harsanyi introduces special assumptions (e.g., a "basic probability distribution" that influences players' priors) that compromise a purely subjective view of probability. Later, Harsanyi [16] suggested a "tracing procedure" for analyzing games that, while Bayesian in some mechanical respects, does not use subjective probability. The procedure looks for criteria by which a player's prior must be of a certain form, thus adopting the "necessitarian" conception of probability (Jeffreys, [20]) rather than the subjectivist position. The work of Boge and Eisele [7] is also relevant. They assume that the opponent's utility function is unknown and that opponents will also act in accordance with Bayes principle. These special assumptions lead to some very difficult mathematics and they do not explore the more general implications of the subjective probability view for game theory. The most closely related work in point of view we have been able to discover is Sanghvi and Sobel [38], [39], who look for the theoretical case in which the players will act in such a way as to leave unchanged each player's probability distribution on the other's action. Of course, this requires many plays of the same game and special assumptions. See also Eliashberg [11], [12].

Basic to the Savage tradition (and the allied work of DeFinetti [9]), a decision-maker has a subjective probability opinion with respect to all of the unknown contingencies affecting his payoffs. In particular in a simultaneous-move two-person game, the player whom we are advising is assumed to have an opinion about the major contingency faced, namely what the opposing player is likely to do. If I think my opponent will choose strategy i ($i = 1, \ldots I$) with probability p_i, I will choose any strategy j maximizing $\sum_{i=1}^{I} p_i u_{ij}$, where u_{ij} is the utility to me of the situation in which my opponent has chosen i and I have chosen j. Since the opponent's utilities are important only in that they affect my views $\{p_i\}$ of what my opponent may do, the distinction between zero and non-zero sum games is much less important in this theory than in von Neumann and Morgenstern's formulation. Dominance, weak or strong, is obviously still important in this theory; if a strategy j exists that maximizes u_{ij} for all i, I will choose j, whatever subjective probabilities, p_i, I assign.

What role would the minimax principle for a zero-sum game play in such a theory? Suppose for example that our opponent announced his intention, and committed himself in an unbreakable contract, to use the minimax strategy. Then there would be, in all games without dominant strategies for both players, several choices j each of which would yield an expected utility equal to the value of the game, and whose mixture with appropriate weights would be my minimax strategy. Choosing any one of these, or the minimax or any other mixture of them would be equally good, and would yield the value of the game to me. Thus, the minimax strategy is not ruled out by the subjective approach, but does not here have the strong probative force given it by von Neumann and Morgenstern.

Minimax theory is, of course, incomplete, in that it does not suggest what I should do if I believe that my opponent is not playing the minimax strategy. The experimental evidence (Rapoport and Orwant [35]; Lieberman [23]) suggests that minimax players are the exception. And if my opponent is not playing the minimax strategy, there will be, in most games, a strategy that I can follow which is superior to minimax.

That minimax strategies are a special case is an illustration of an important general connection between the subjective theory of games and the von Neumann-Morgenstern view, namely *solution concepts are a basis for particular prior distributions*.[2] This helps to explain the difficulty in non-zero sum, N-person game theory of finding an adequate solution concept: no single prior distribution is likely to be adequate to all players and all situations in such games.

Multiplayer games present no essentially new conceptual difficulties. To decide on an optimal strategy, a player needs to know his probability, $p_{i_1, i_2 \ldots i_k}$, that player 1 will choose i_1, player 2 strategy i_2, etc., then the optimal strategy is to choose a strategy j that maximizes $\sum p_{i_1, i_2 \ldots i_k} u_{i_1, i_2 \ldots i_k j}$ where the summation extends over all strategies available to the other players. Such games may, however, pose important new information collection and processing difficulties. It becomes more costly, if not infeasible, to acquire firm priors on the behavior of many players.[3]

Similarly games studied by an outside observer are different from the point of view of a player. Thus, an outside observer will have a probability distribution on the priors of the players of the game. The analysis from the viewpoint of an observer will be somewhat similar to the analysis of Lindley, Tversky and Brown (1980) on the elicitation problem from the viewpoint of an external person.

A possible problem with the theory advocated here is the infinite regress.[4] If he thinks I think he'll do x, then he'll do y. If he thinks I think he thinks I think he'll do y, etc. It is true that a subjectivist Bayesian will have an opinion not only on his opponent's behavior, but also on his opponent's belief about his own behavior, his opponent's belief about his belief about his opponent's behavior, etc. (He also has opinions about the phase of the moon, tomorrow's weather and the winner of the next Superbowl). However, in a single-play game, all aspects of his opinion except his opinon about his opponent's behavior are irrelevant, and can be ignored in the analysis by integrating them out of the joint opinion.

[2] Most "solution concepts" are from the perspective of an external observer of games. For the observer, the solution concept is the basis of a prior about the game's outcome. At the level of the individual player, assumptions about opponents' behavior correspond to the individual's priors. In minimax, the assumption is that the opponent will certainly do his best assuming that I will do my best with full information; each player is assumed to believe that the other is sure to play his minimax strategy.

[3] A common analytic response to the informational difficulties arising from the multiplayer circumstance is to treat the other players as an undifferentiated mass behaving in an analytically tractable fashion

> . . . in market environments it is assumed, at least implicitly, that there are many agents. In such contexts then the reward to a single agent depends not only on his own decision but also on the decisions of other agents. Thus, to predict the decisions which agents will make in a multi-agent environment there is needed some notion of consistency. We emphasize here as did Rothschild [37] that *the things which each agent takes as given in making his own decision must be consistent with maximizing behavior on the part of the other agents*. (Prescott and Townsend, [33, p. 2]).

This sort of simplification strategy would be useful to a player with no basis for differentiating among other agents and a firm prior on the decision mechanism of other agents (e.g. "maximizing behavior") as an accurate predictive theory of their behavior.

[4] Harsanyi [15, p. 163] speculates that:

> . . . the basic reason why the theory of games with incomplete information has made so little progress so far lies in the fact that these games give rise, or at least appear to give rise, to an infinite regress in reciprocal expectations on the part of the players.

Also see Riker and Ordeshook, [36] and Young [52, p. 28] for other statements on the perceived importance of the "infinite regress problem."

Multiple-play games do introduce a new complication in the Bayesian theory. To help fix ideas, we will take a two game sequence in which game 2 is played, and then game 1. (We use reverse numbering because backwards induction, an essential part of the reasoning for this case, makes it convenient.) When only game 1 is left, only my opinion about my opponent's action in game 1 is relevant. In thinking about my action in game 2, I must take into account not only my opinion about what my opponent is likely to do in game 2, but also my opinion about the effect my strategey in game 2 may have on my opponent's strategy in game 1, conditional on each of the actions I may take in game 2. It is this feature that makes multiple play games very interesting from the subjectivist Bayesian perspective. This also suggests that certain multiple-play games, such as the prisoner's dilemma, may be illuminated by this approach, since it is precisely the effect of today's decision on the actions of others, tomorrow, that has caused so much debate (see Rapoport and Chammah [34] and Axelrod [5]).

It is notable that, although Bayesian theory is basically prescriptive (Savage, [40] and DeFinetti [9]) predictive theories are not neglected. Recently a line of psychological investigation has suggested that human beings find it quite difficult to meet the coherence requirements of various rational decision prescriptions, including Bayesian theory (for reviews see, Tversky and Kahneman [47], [48]; Hogarth, [19], Slovic *et al.* [45]; and Nisbett *et al.* [30]). These observations in turn have led to questions about whether the normative approach of standard Bayesian theory, or an alternative model of human cognition that fits the facts better, is more deserving of scientific attention. As one of the players, or as an advisor to one of the players, the Bayesian axioms of Savage are prescriptively compelling and consequently we would seek to play in accordance with them. This requires the best predictive theory we can find of the likely actions of the other player(s). Thus, both the prescriptive and predictive theories appear to have natural roles; neither need or should be chosen to the exclusion of the other. It may or may not be the case that the best prescriptive theory and the best predictive theory are one in the same in any given instance. This is an empirical question. As competitors, we seek to profit from whatever peculiarities we find in the play of our opponent(s).

Finally, there is the question of where all these prior distributions "come from." The experimental literature on elicitation of prior beliefs in general is indeed meagre. One line, pursued especially by Savage [41] tries to obtain elicitations of priors (and utilities) as optimal behavior under certain specified conditions. This literature is very close to material on proper scoring rules (Brier, [8]). A second line is more heuristic, trying to find questions easily answered by the subject and still transformable into the information sought. A recent paper along these lines is Kadane, *et al.* [21]. Finally, there is the paper of Lindley, Tversky and Brown [24] which takes a Bayesian approach to elicitation itself, although with very special models. None of this literature on elicitation has, to our knowledge, dealt with beliefs in a game context. This is a potentially fruitful area for further research.

In this paper we have explored the consequences for game theory of adopting a subjective view of probability. The consequences are large. Distinctions that appear to be important in von Neumann and Morgenstern (two-person vs. $n>2$ person, zero-sum vs. variable sum) appear not to be critical in this formulation. However, the distinction between single play and repeated play games seems more important than in the original von Neumann and Morgenstern work.

The Bayesian view of games clarifies the proper, respective roles of prescriptive and predictive theory. The view also raises some fundamental questions about the value of pursuing solutions to games, solutions that presume symmetrical behavior in the two-player case and homogeneous behavior for all players in the multi-player case.

> . . . the achievement of determinate solutions for two person, non-zero-sum games through the estimation of subjective probabilities requires the introduction of an assumption to the effect that the individual employs some specified rules of thumb in assigning probabilities to the choices of the other player. But this is not a very satisfactory position to adopt within the framework of the theory of games. Logically speaking, there is an infinite variety of rules of thumb that could be used in assigning subjective probabilities, and game theory offers no persuasive reason to select any one of these rules over the others. This problem can be handled by introducing new assumptions (or empirical premises) about such things as the personality traits of the players. But such a course would carry the analyst far outside the basic structure of the theory of games, requiring a fundamental revision of the basic perspective of game theory. (Young, [52, pp. 28–29]).

From the subjectivist Bayesian perspective, game theorists are already "(employing) some specified rules-of-thumb in assigning probabilities to the choices of the other player." Assuming that your opponent will play a minimax strategy which you attempt to construct from his perspective given information in any particular game about his pay-offs and preferences is an example of such a "rule-of-thumb." At best, this rule-of-thumb is a partial basis for forming your prior about your opponent's likely behavior in certain simple game situations. It is not a logically compelling prescription for your own play (Ellsberg, [13]). And it is not a very efficient predictive theory for most games (Rapoport and Orwant, [35]).

Perhaps it is time to reunite the two streams of work descended from von Neumann and Morgenstern (1944), prescriptive theories of individual decisionmaking and theories of strategically interactive decisions, and to look to other disciplines such as cognitive psychology for predictive theories of decisional behavior.

References

1. Aumann, R. J., "Mixed and Behavior Strategies In Infinite Extensive Games," *Advances in Game Theory*, Princeton Univ. Press, Princeton, N.J., 1964, pp. 627–650.
2. ———, "Some Thoughts On The Theory of Cooperative Games," *Advances In Game Theory*, Princeton Univ. Press, Princeton, N.J., 1964, pp. 407–442.
3. ——— and Maschler, M., "Repeated Games with Incomplete Information, The Zero-Sum Extensive Case," Report to the U.S. Arms Control and Disarmament Agency, Washington D.C. Final Report on Contract ACDA/ST-143, prepared by *Mathematica*, Princeton, 1968, Chapter III, pp. 25–108.
4. ——— and Shapley, L., *Values of Non-Atomic Games*, Princeton Univ. Press, Princeton, N.J., 1974.
5. Axelrod, R., "Evolution of Cooperation in the Prisoner's Dilemma," Institute of Public Policy Studies, University of Michigan, 1979.
6. Blackwell, D. and Girshich, M. A., *Theory of Games and Statistical Decisions*, Wiley, New York, 1959.
7. Boge, W. and Eisele, T., "On Solutions of Bayesian Games," *Internat. J. Game Theory*, Vol. 8 (1979), pp. 193–215.
8. Brier, G. W., "Verification of Forecasts Expressed in Terms of Probability," *Monthly Weather Rev.*, Vol. 75 (1950), pp. 1–3.
9. DeFinetti, B., *Theory of Probability* (2 vols.), Wiley, New York, 1974.
10. DeGroot, M. H., *Probability and Statistics*, Addison-Wesley, Reading, Mass., 1975.
11. Eliashberg, J., "Bayesian Analysis of Competitive Decision-Making Situations," unpublished mimeo., 1980.
12. ———, "An Investigation of Competitive Preference Structures and Posterior Performance Through a Bayesian Decision-Theoretic Approach," 1980.
13. Ellsberg, D., "Theory of the Reluctant Duelist," reprinted in [52].
14. Fienberg, S. E. and Zellner, A., eds., *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, North-Holland, Amsterdam, 1975.
15. Harsanyi, J., "Games with Incomplete Information Played by 'Bayesian' Players," *Management Sci.*, Vol. 14, No. 3 (1967), pp. 159–182; Vol. 14, No. 5 (1968), pp. 320–334; Vol. 14, No. 7 (1968), pp. 486–502.
16. ———, "The Tracing Procedure: A Bayesian Approach to Defining a Solution for *n*-person Noncooperative Games," *Internat. J. Game Theory*, Vol. 4, No. 2 (1975), pp. 61–94.
17. ———, "Bargaining in Ignorance of the Opponent's Utility Function," *J. Conflict Resolution*, Vol. 6 (1962), pp. 29–38.

18. HARSANYI, J. AND SELTEN, R., "A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information," *Management Sci.*, Vol. 18, No. 5 (1972), pp. 80–106.
19. HOGARTH, R. M. "Cognitive Processes and the Assessment of Subjective Probability Distributions," *J. Amer. Statist. Assoc.*, Vol. 70, No. 350 (1975).
20. JEFFREYS, H., *Theory of Probability*, (3rd edition), Oxford Univ. Press, London, 1961.
21. KADANE, J. B., DICKEY, J. M., WINKLER, R. L., SMITH, W. S. AND PETERS, W. C., "Interactive Elicitation of Opinion for a Normal Linear Model," *J. Amer. Statist. Assoc.*, Vol. 75 (1980), pp. 845–854.
22. KOHLBERG, E., "Optimal Strategies in Repeated Games with Incomplete Information," *Internat. J. Game Theory*, Vol. 4 (1975), pp. 7–24.
23. LIEBERMAN, B., "Experimental Studies of Conflict in Some Two-and Three-Person Games," in Massarik, F. and Ratoosh, P. (eds.), *Mathematical Explorations in Behavioral Science*, Irwin, Homewood, Ill., 1965.
24. LINDLEY, D., TVERSKY, A. AND BROWN, R., "On the Reconciling Probability Assessments," *JRSS*, Series A (1980).
25. LINDLEY, D., "Bayesian Statistics: A Review," SIAM, Philadelphia, Penn., 1971.
26. LUCAS, W. F., "An Overview of the Mathematical Theory of Games," *Management Sci.*, Vol. 18, No. 5 (1972).
27. MCKELVEY, R. D. AND ORDESHOOK, P. C., "An Experimental Test of Solution Theories for Cooperative Games in Normal Form," manuscript, 1980.
28. MERTENS, J. F. AND ZAMIR, S., "The Value of Two-Person Zero-Sum Repeated Games with Lack of Information on Both Sides," *Internat. J. Game Theory*, Vol. 1 (1971), pp. 39–64.
29. MILLER, G. R. AND STEINFATT, T. M. "Communication in Game Theoretic Models of Conflict," in Miller, G. R. and Simons, H. W. (eds.), *Perspectives on Communication in Social Conflict*, Prentice-Hall, Englewood Cliffs, N.J., 1974.
30. NISBETT, R. E., CRANDALL, R., BORGIDA, E. AND REED, H., "Popular Induction: Information is Not Necessarily Informative," in *Cognition and Social Behavior*, Lee W. Gregg, editor, Lawrence Erlbaum Assoc., Hillsdale, 1976.
31. PONSSARD, J. P. AND ZAMIR, S., "Zero-Sum Sequential Games with Incomplete Information," *Internat. J. Game Theory*, Vol. 2 (1973), pp. 99–107.
32. ———, "Zero-Sum Games with 'Almost' Perfect Information," *Management Sci.*, Vol. 21, No. 7 (1975), pp. 794–805.
33. PRESCOTT, E. C. AND TOWNSEND, R. M., "Equilibrium Under Uncertainty: Multi-Agent Statistical Decision Theory," to appear in [54].
34. RAPOPORT, A. AND CHAMMAH, A. M., *Prisoner's Dilemma*, Ann Arbor, Mich., 1965.
35. ——— AND ORWANT, C., "Experimental Games: A Review," *Behavioral Sci.*, Vol. 7 (1962), pp. 1–37.
36. RIKER, W. H. AND ORDESHOOK, P. C., *An Introduction to Positive Political Theory*, Prentice-Hall, Englewood Cliffs, N.J., 1973.
37. ROTHSCHILD, M., "Models of Market Organization with Imperfect Information," *J. Political Economy*, Vol. 81 (1973), pp. 1283–1308.
38. SANGHVI, A. P. AND SOBEL, M. J., "Bayesian Games as Stochastic Processes," *Internat. J. of Game Theory*, Vol. 5, No. 1 (1976), pp. 1–22.
39. ——— AND ———, "Sequential Games as Stochastic Processes," *Stochastic Processes Their Appl.*, Vol. 6 (1978), pp. 323–336.
40. SAVAGE, L. J., *Foundations of Statistics*, Wiley, New York, 1954.
41. ———, "Elicitation of Personal Probabilities and Expectations," *J. Amer. Statist. Assoc.*, Vol. 66 (1971), pp. 783–801.
42. SCHOTTER, A. AND SCHWODIAUER, G., "Economics and Game Theory: A Survey," *J. Economic Literature* (June 1980).
43. SHUBIK, J., "Some Experimental Non-Zero Sum Games with Lack of Information About the Rules," *Management Sci.*, Vol. 8, No. 2 (1962), pp. 215–234.
44. SIMON, H. A., "A Comparison of Game Theory and Learning Theory," *Models of Man*, H. A. Simon, Wiley, New York, 1957, pp. 274–279.
45. SLOVIC, P., FISCHOFF, B. AND LICHTENSTEIN, S., "Behavioral Decision Theory," *Annual Rev. Phych.*, Vol. 28 (1977), pp. 1–39.
46. SORIN, S., "A Note on the Value of Zero-Sum Sequential Repeated Games with Incomplete Information," *Internat. J. Game Theory*, Vol. 8, No. 4 (1979), pp. 217–223.
47. TVERSKY, A. AND KAHNEMAN, D., "Judgment Under Uncertainty: Heuristics and Biases," *Science*, Vol. 185 (1974), pp. 1124–1131.
48. ——— AND ———, "The Framing of Decisions and the Psychology of Choice," *Science*, Vol. 211 (1980), pp. 453–458.
49. VON NEUMANN, J. AND MORGENSTERN, O., *The Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, N.J., 1944.

50. VON NEUMANN, J., "On the Theory of Games of Strategy," *Contributions to the Theory of Games*, Princeton Univ. Press, Princeton, N.J., pp. 13–42.
51. WALD, A., *Statistical Decision Functions*, Wiley, New York, 1950.
52. YOUNG, ORAN R., *Bargaining: Formal Theories of Negotiation*, Univ. of Illinois Press, Chicago, Ill., 1975.
53. ZAMIR, S., "On the Relation Between Finitely and Infinitely Repeated Games with Incomplete Information," *Internat. J. Game Theory*, Vol. 1, No. 1 (1971), pp. 179–198.
54. ZELLNER, A., ed., *Bayesian Analysis in Econometrics and Statistics, Essays in Honor of Harold Jeffreys*, North-Holland, Amsterdam, 1980.

COMMENT*

MANAGEMENT SCIENCE
Vol. 28, No. 2, February 1982
Printed in U.S.A.

SUBJECTIVE PROBABILITY AND THE THEORY OF GAMES: COMMENTS ON KADANE AND LARKEY'S PAPER†

JOHN C. HARSANYI‡

The normative solution concepts of game theory try to provide a clear mathematical characterization of what it means to act rationally in a game where all players expect each other to act rationally. Kadane and Larkey reject the use of these normative solution concepts. Yet, this amounts to throwing away an important piece of information to the effect that the players are rational and expect each other to be rational. Even in situations where the players do not expect each other to act with complete rationality, normative game theory can help them heuristically to formulate reasonable expectations about the other players' behavior.
(GAME THEORY; RATIONALITY; BAYESIANISM; SUBJECTIVE PROBABILITIES)

According to some textbooks, there are two versions of Bayesian decision theory. The *subjectivist* version supposedly permits the decision-maker to choose his subjective probabilities in any arbitrary way (at least as long as they obey the addition and the multiplication laws of the probability calculus). In contrast, the *necessitarian* version uniquely specifies the subjective probabilities which a rational decision-maker can use in any given situation.

A little reflection will show that both textbook versions of Bayesian theory are empty caricatures. Admittedly, the verbal pronouncements of some Bayesian statisticians often come quite close to one or the other of these two extreme views, but I have yet to see a working statistician whose statistical practice has actually been governed by either view. When confronted with real-life statistical problems, all competent Bayesian statisticians will recognize that in some situations there is only *one* rational prior distribution, whereas in other situations we have a more or less free choice among many *alternative* priors.

Even Leonard Savage, whose views come closest to extreme subjectivism among distinguished Bayesian statisticians, has always admitted that in some situations (viz., those involving random devices with suitable physical symmetries, such as fair coins or fair dice, etc.) *all* reasonable people will use the *same* (rectangular) probability distributions. More recently, many Bayesian physicists have argued convincingly that in thermodynamics and in some other branches of physics one can derive the classical

*This Comment has been refereed.
†Accepted by Ambar G. Rao; received June 1981.
‡University of California, Berkeley.

0025-1909/82/2802/0 $01.25