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WHERE LUCE AND KRANTZ DO REALLY GENERALIZE SAVAGE'S DECISION MODEL

1. INTRODUCTION

In this paper I shall somewhat investigate several formulations of decision theory from a purely quantitative point of view, thus leaving aside the whole question of measurement. Since almost any foundational work on decision theory strives at proving nicer and nicer measurement results and representation theorems, I feel obliged to give a short explanation of my self-imposed limitation. The first and best reason for it is that I have not got anything new to say about measurement, and the second is that one need not say anything:

It was hard work to convince economists that cardinalization is possible and meaningful. This was accomplished by proving existence and uniqueness theorems establishing the existence of cardinal functions (e.g. subjective utilities and probabilities) unique up to certain transformations that mirror ordinal concepts (e.g. subjective preferences) in a certain way. And surely, such theorems provide an excellent justification for the use of cardinal concepts. The eagerness in the search for representation theorems, however, is not really understandable but on the supposition that they are the only justification of cardinal concepts, and this assumption is merely a rather dubious conjecture. After all, philosophers of science have been debating about theoretical concepts for at least 40 years, and, though the last word has not yet been spoken, they generally agree that it is possible to have meaningful, yet observationally undefinable theoretical notions.[1] And the concepts of subjective probability and utility *are* theoretical notions of decision theory. Thus if philosophers of science are right, they need not necessarily be proved observationally definable by representation theorems for being meaningful.[2]

For that reason I consider quantitative decision models fundamental for decision theory and measurement as part of the confirmation or testing theory of the quantitative models. Of course, the latter is important for evaluating the former, but there may be different (e.g. conceptual) grounds for finding one quantitative decision model more satisfac-

tory than another. And this is all the more the case, if we confine ourselves, as we will do, to comparing decision models which agree on the core of decision theory by assuming the principle of maximizing expected utility. So, if one is convinced that this principle is essentially sound, it seems to be good strategy first to formulate quantitative decision model as satisfactory as possible that encorporates this principle and then to explore its empirical consequences.

In this spirit, then, I will somewhat investigate the quantitative part of the, hopefully, most important variants of decision theory, namely those of Savage [12], Fishburn [5], Jeffrey [7], and Luce and Krantz [10]. As the title of the paper suggests, I will especially dwell upon the fourth, perhaps least understood variant.

2. PROBABILITIES FOR ACTS

Before focussing attention on these four variants I will introduce a general decision theoretic principle that almost everyone complies with and finds so obvious that it scarcely needs mentioning, let alone justification. But since our evaluation of Jeffrey's decision model and that of Luce and Krantz rests essentially upon this principle, I will state it explicitly and provide some reasons for it which are independent of any special conception of decision theory. The principle says: *Any adequate quantitative decision model must not explicitly or implicitly contain any subjective probabilities for acts* (except, perhaps, trivial conditional probabilities like $P(H|H) = 1$ for some act H or $P(H|H') = 0$ for two disjoint acts H and H').

Note that this principle requires acts to be things which are under complete control of the decision maker – though not in an absolute sense, but only relative to the decision model purporting to describe the decision maker. For instance, going from here to there is not under complete control of the decision maker in an absolute sense; there may be hindrances. But for the sake of simplicity, a decision model may take it to be under the agent's complete control by just not worrying about possible hindrances. If, however, the decision model would consider all the individual steps, going from here to there could not function as an act in the model, because the decision maker could have subjective probabilities for it conditional on various sequences of steps.

Now, probably anyone will find it absurd to assume that someone has subjective probabilities for things which are under his control and which he can actualize as he pleases. I think this feeling of absurdity can be converted into more serious arguments for our principle:

First, probabilities for acts play no role in decision making. For, what only matters in a decision situation is how much the decision maker likes the various acts available to him, and relevant to this, in turn, is what he believes to result from the various acts and how much he likes these results. At no place does there enter any subjective probability for an act. The decision maker chooses the act he likes most – be its probability as it may. But if this is so, there is no sense in imputing probabilities for acts to the decision maker. For one could tell neither from his actual choices nor from his preferences what they are. Now, decision models are designed to capture just the decision maker's cognitive and motivational dispositions expressed by subjective probabilities and utilities which manifest themselves in and can be guessed from his choices and preferences. Probabilities for acts, if they exist at all, are not of this sort, as just seen, and should therefore not be contained in decision models.

The strangeness of probabilities for acts can also be brought out by a more concrete argument: It is generally acknowledged that subjective probabilities manifest themselves in the readiness to accept bets with appropriate betting odds and small stakes. Hence, a probability for an act should manifest itself in the readiness to accept a bet on that act, if the betting odds are high enough. Of course, this is not the case. The agent's readiness to accept a bet on an act does not depend on the betting odds, but only on his gain. If the gain is high enough to put this act on the top of his preference order of acts, he will accept it, and if not, not. The stake of the agent is of no relevance whatsoever.

One might object that we often do speak of probabilities for acts. For instance, I might say: "It's very unlikely that I shall wear my shorts outdoors next winter." But I do not think that such an utterance expresses a genuine probability for an act; rather I would construe this utterance as expressing that I find it very unlikely to get into a decision situation next winter in which it would be best to wear my shorts outdoors, i.e. that I find it very unlikely that it will be warmer than 20°C next winter, that someone will offer me DM 1000. – for wearing shorts outdoors, or that fashion suddenly will prescribe wearing shorts, etc. Besides, it is characteristic of

such utterances that they refer only to acts which one has not yet to decide upon. As soon as I have to make up my mind whether to wear my shorts outdoors or not, my utterance is out of place.

It is not necessary now to discuss our principle any further, though it would be interesting to do so. In fact, I believe that it touches upon and is essential to very fundamental matters such as the concept of an act, the Newcomb paradox and our views of causality and, in effect, the whole field centering about freedom of will.[3] But even without embedding the principle into a broader coherent picture, I hope I have given it support firm enough to make its consequences for decision theory acceptable even to those who had some doubt about it.

I have still to note an immediate consequence of our principle that will prove important: If we do not allow probabilities for acts, we cannot allow unconditional probabilities for act-dependent events, either. For, if a certain event, A, is act-dependent, i.e. if $P(A|H) \neq P(A|\bar{H})$ for some act H and its complement $\bar{H}$, we could infer $P(H)$ from $P(A)$, since $P(H) = P(A) - P(A|\bar{H})/P(A|H) - P(A|\bar{H})$. Thus, if a decision model ascribes an unconditional subjective probability for an event to an agent, it thereby assumes that the agent considers this event to be independent of his acts.

After these preliminaries we can turn to our main subject, the discussion of quantitative decision models founded on the principle of maximizing expected utility.

3. SAVAGE'S DECISION MODEL

Without doubt, Savage's variant of decision theory is the most prevailing one and its structure is well known. It consists of a set Ω of possible world states specifying how the circumstances relevant to the decision at hand might be; a set C of possible consequences which might result from the acts available to the decision maker; for mathematical convenience and for reasons of measurability, a σ-algebra $\mathscr{A}$ of events over Ω and a σ-algebra $\mathscr{C}$ over C; a set $\mathscr{F}$ of available acts; a function P on $\mathscr{A}$ specifying the decision maker's subjective probabilities for events; a function V on C specifying the decision maker's subjective utilities for consequences; and a function U on $\mathscr{F}$ specifying the expected utilities of acts. The characteristic feature of the Savage model is that it assumes the decision maker to associate exactly one consequence with each world

state and act, and for that reason acts can be formalized as functions from the set of world states into the set of consequences. Thus we may define:

DEFINITION 1: $\langle \Omega, \mathcal{A}, C, \mathcal{C}, \mathcal{F}, P, V, U \rangle$ is a *Savage-model* iff
(1) $\langle \Omega, \mathcal{A}, P \rangle$ is a σ-additive probability space,
(2) $\langle C, \mathcal{C} \rangle$ is a measurable space,
(3) $\mathcal{F}$ is a non-empty set of $\mathcal{A}$-$\mathcal{C}$-measurable functions from Ω into C,[4]
(4) V is a $\mathcal{C}$-measurable function from C into the reals,
(5) for every $f \in \mathcal{F}$, $V \circ f$ is P-integrable,
(6) U is the function from $\mathcal{F}$ into the reals for which $U(f) = \int V \circ f \, dP$ for every $f \in \mathcal{F}$.[5]

The defects of Savage-models are also well known, and I will summarize them in short. The representation of decision situations by Savage-models is restrictive in two ways:

(1) The first restriction lies in the representation of the decision maker's motivational dispositions by a utility function solely defined for consequences. This is because the decision maker might also assess possible world states with utility.[6] There are two ways of getting around this objection. First, one could conceive possible world states as being components of possible consequences. Then the objection no longer applies. But this is not a nice solution, since it leads to a certainly unintended double representation of world states in Savage-models: first as world states proper and then as components of possible consequences. Secondly, one could argue that the decision maker's utilities for world states are not relevant to his decision. For the world states are conceived as being act-independent; there is nothing the decision maker can do about them. Therefore utilities for world states at most contribute a negligible constant to the expected utility of acts, and we can simply forget about them. But this argument is fallacious; it tacitly assumes world states and consequences to be utility-independent. Only then do utilities for world states add a negligible constant to the expected utilities of acts. So I take this to be the first restriction of Savage-models: They assume utility-independence between world states and consequences.

(2) The more severe second restriction lies in the representation of the decision maker's cognitive dispositions by a probability measure solely defined for events. But that is not a fully correct statement of the

restriction; there are hidden act-conditional probabilities for conse-
quences in Savage-models. For by representing an act f as a function from
world states into consequences, Savage-models impute to the decision
maker that he is sure that the consequence $f(\omega)$ will result when world
state ω obtains and act f is chosen, or, in loose probabilistic terms, that the
subjective probability of $f(\omega)$ conditional on f and ω is 1. And the image
P_f of the measure P relative to $f \in \mathcal{F}$ defined by $P_f(D) = P(f^{-1}(D))$ for
every $D \in \mathcal{C}$ just expresses the decision maker's subjective probabilities
for consequence-events conditional on the act f. Note that the expected
utility $U(f)$ of an act f can then be written as $U(f) = \int V \, dP_f$.

Thus, more correctly, the second restriction lies in the rigid connection
between world states and acts on the one hand, and consequences on the
other hand, which results from representing acts by functions, or other-
wise put: in the assumption that all act-conditional probability measures
P_f for consequences can be represented as images of one single probabil-
ity measure P for world states. And clearly, the decision maker's subjec-
tive probabilities need not be of this special structure. There are many
simple decision situations in which it is difficult, if not impossible to
identify appropriate world states.[7]

Savage himself clearly recognized these difficulties and summarized
them by asking:

> Is it good, or even possible, to insist, as this preference theory does, on a usage in which
> acts are without influence on events and events without influence on well-being? ([13], p.
> 307f)

4. FISHBURN'S DECISION MODEL

Fishburn [5], Chapters 2 and 3, has proposed a very simple and natural
way of overcoming both restrictions built in into Savage-models at one
stroke. Just merge world states into consequences and let the decision
maker have arbitrary act-conditional probabilities for the new conse-
quences.[8] Formally this reads:

DEFINITION 2: $\langle C, \mathcal{C}, \mathcal{F}, P, V, U \rangle$ is a *Fishburn-model* iff
(1) $\langle C, \mathcal{C} \rangle$ is a measurable space,
(2) $\mathcal{F}$ is a non-empty set,
(3) P is a family of probability measures $P_f(f \in \mathcal{F})$ on $\mathcal{C}$ indexed by $\mathcal{F}$,

(4) V is a function from C into the reals that is P_f-integrable for every $f \in \mathscr{F}$,

(5) U is the function from $\mathscr{F}$ into the reals for which $U(f) = \int V \, dP_f$ for every $f \in \mathscr{F}$.

Of course, Savage-models can be embedded into Fishburn-models: Let $\langle \Omega, \mathscr{A}, C, \mathscr{C}, \mathscr{F}, P, V, U \rangle$ be a Savage-model. Set $C' = \Omega \times C$, $\mathscr{C}'$ equal to the σ-algebra over C' generated by $\{A \times D \mid A \in \mathscr{A}, D \in \mathscr{C}\}$, and $\mathscr{F}' = \mathscr{F}$. Let for each $f \in \mathscr{F}$ P'_f be the probability measure on $\mathscr{C}'$ for which $P'_f(A \times D) = P(A \cap f^{-1}(D))$ for every $A \in \mathscr{A}$ and $D \in \mathscr{C}$,[9] and let P' be the family of all P'_f $(f \in \mathscr{F}')$. Finally, let V' be defined on C' by $V'(\omega, c) = V(c)$ for every $\omega \in \Omega$ and $c \in C$, and put $U' = U$. Then, obviously, $\langle C', \mathscr{C}', \mathscr{F}', P', V', U' \rangle$ is a Fishburn-model.

This embedding demonstrates particularly clearly in which ways Savage-models are restricted and how Fishburn generalizes them. For neither in actual life nor in Fishburn-models need utilities and probabilities to have the special structure of the V' and P' of the above embedding.

Fishburn [5], pp. 53ff, has pointed out, however, that the reverse embedding is also possible by defining the Savage-set of possible world states as the set of all functions from the Fishburn-set of acts into the Fishburn-set of consequences.[10] But this embedding is not a natural one, since these functions must be interpreted as conjunctions of subjunctive conditionals saying "if act f_1 were chosen, consequence c_i would come about, and if act f_2 were chosen, consequence c_j would come about, and ...". I think we had better leave subjunctive conditionals and probabilities for them alone.[11]

It seems therefore justified to say that Fishburn-models are a genuine generalization of Savage-models. Moreover, they are conceptually simpler. Indeed, I think that Fishburn-models are essentially the most natural and satisfactory quantitative decision models, and I suppose that everybody would happily concede this if they could forget about the indispensability of representation theorems; metrization of Fishburn-models seems to be a hopeless task. Constructing a utility function and a probability measure from preferences already looked like drawing a rabbit out of the hat, but to construct a utility function and a whole family of probability measures from preferences would mean pulling an elephant out of the hat.

Balch and Fishburn [1] even managed the latter trick, by using a very big hat, i.e. extraneous probabilities. And this constitutes an essentially weaker result, since the only thing that matters for decisions and preferences are the decision maker's *subjective* probabilities. Therefore, the use of extraneous, *objective* probabilities only makes sense on the assumption that the decision maker's subjective probabilities are the same as the objective probabilities. Thus, metrizations using extraneous probabilities are in fact starting with a large number of imputed subjective probabilities, thereby failing to be fundamental.

But from a purely quantitative point of view Fishburn-models seem to do an optimal job. Let us see whether we can gain new insights from later developed variants of decision theory.

5. JEFFREY'S DECISION MODEL

Jeffrey was not at all happy with Savage's strict separation of possible world states carrying subjective probabilities on the one hand and possible consequences carrying subjective utilities in the other hand. After all, cognitive and motivational dispositions refer to the same things, namely propositions or states of affairs. E.g. we might wish as well as believe that it will rain tomorrow. Thus Jeffrey proposed a holistic variant of decision theory, as he calls it, which operates with only one big σ-algebra of events comprising world states, consequences, and even acts as special events. Of course, these acts have to form a partition of the sure event. Subjective probabilities and utilities are then defined for these events.[12] It must be remarked, however, that these utilities already express *expected* utilities of the events. They must therefore satisfy a certain averaging condition stated below. Again for mathematical convenience, the following definition considers the σ-additive case:

DEFINITION 3: $\langle \Omega, \mathcal{A}, \mathcal{F}, P, U \rangle$ is a *Jeffrey-model* iff
(1) $\langle \Omega, \mathcal{A}, P \rangle$ is a σ-additive probability space,
(2) $\mathcal{F}$ is a $\mathcal{A}$-measurable partition of Ω,
(3) U is a function from $\mathcal{A}$-$\{\varnothing\}$ into the reals such that the following holds: If $(A_i)_{i \in I}$ is a countable family of mutually disjoint, non-empty events from $\mathcal{A}$, $A = \bigcup_{i \in I} A_i$ and $P(A) \neq 0$, then

$$(5.1) \qquad U(A) = \sum_{i \in I} U(A_i) \cdot P(A_i | A) .$$

Unlike other models, Jeffrey-models do not contain a function V expressing absolute or non-expected utilities. But there is no real difference here, as the following consideration shows:[13] Let $\langle \Omega, \mathcal{A}, \mathcal{F}, P, U \rangle$ be a Jeffrey-model. Define μ on $\mathcal{A}$ by $\mu(\varnothing) = 0$ and $\mu(A) = U(A) \cdot P(A)$. Because of (5.1) μ obviously is a signed measure on $\mathcal{A}$ which is absolutely continuous with respect to P. Applying the Radon–Nykodym theorem for signed measures[14] to that situation we get: There is an almost uniquely determined $\mathcal{A}$-measurable function V from Ω into the reals such that for each nonnull $A \in \mathcal{A}$:

$$(5.2) \qquad U(A) = \frac{1}{P(A)} \cdot \int_A V \, dP.$$

And, of course, any function of the form (5.2) satisfies (5.1). Thus there is a sort of almost-one-to-one-correspondence between expected utility functions on $\mathcal{A}$ and absolute utility functions on Ω.

Are Jeffrey-models acceptable decision models? According to our principle from Section 2, the answer is obviously no: acts are part of the σ-algebra of events for which P is defined, and therefore Jeffrey-models contain probabilities for acts.[15]

What can we do about this situation? Removing acts from the σ-algebra $\mathcal{A}$ and considering the events in $\mathcal{A}$ as act-independent is no solution. For then we would only have acts and world states in the sense of Savage, but nothing that could give us expected utilities for acts. Moreover, the appealing holistic character of Jeffrey-models would be lost.

Rather, we must remove probabilities for acts and unconditional probabilities for act-dependent events from Jeffrey-models. As can easily be conceived, however, the resulting model would differ from Fishburn-models only in minor points. It would be a bit more holistic than Fishburn-models already are (which contain utilities as well as act-conditional probabilities for consequences); for in the resulting model acts would still be part of one big σ-algebra of propositions or events. And it would be slightly more general than Fishburn-models in being able to deal with intrinsic utilities for acts (cf. Notes 6 and 8).[16]

Note also that the expected utility function U of a Jeffrey-model would be affected by such a modification in the representation of subjective probabilities. If we eliminate all the unwelcome probabilities from a

Jeffrey-model and adopt a Fishburn-like probability structure, then there would be some events, e.g. unions of acts, to which one could not meaningfully assign expected utilities. The standard procedure of having both a function expressing absolute, non-expected utilities and an expected utility function for acts seems therefore preferable.[17]

6. CONDITIONAL DECISIONS

Finally we will examine the variant of decision theory which Luce and Krantz [10] have proposed and for which they have proved a representation theorem (in [8], Chapter 8). This variant is all the more interesting because Luce and Krantz explicitly refer to Fishburn's suggestions ([10], p. 253); thus there is some hope that they have solved the metrization problem for a quantitative decision model equally general as the Fishburn model.

This hope is dampened by the fact that at first sight their quantitative decision model looks similar to Savage's. Yet there are two novelties. The first consists in a new formal representation of the acts available to the decision maker and the second brings about a more general treatment of utilities. Ultimately both novelties are connected, but our discussion will be clearer if we separate them. So this section is devoted to the first novelty and the next to the second.

The starting point of Luce and Krantz is the same as that of Fishburn. They, too, are dissatisfied with Savage's world states being act-independent. And they trace the cause of this drawback to the fact that in Savage-models acts are represented by functions defined for *all* world states. This representation does not account for the fact that acts may determine at least partially which events occur. For instance, if one decides to take the plane it is utterly impossible to have a car crash.

Consequently, Luce and Krantz change Savage-models in the following way: Like Savage they assume a set Ω of world states, a σ-algebra $\mathscr{A}$ of events over Ω, and a σ-additive probability measure P on $\mathscr{A}$ expressing the decision maker's subjective probabilities.[18] Further, they assume a set C of consequences to which I will add a σ-algebra $\mathscr{C}$ over C on grounds of measurability. Finally, I shall for the moment disregard Luce's and Krantz' second novelty and assume a Savage-like utility function V on C.

The crucial change against Savage lies in the representation of acts. To make allowance for the influence of acts on the events happening, Luce and Krantz represent acts by *partial* functions defined only for some, not necessarily for all world states. The idea behind this is the following: If an act has an influence on the events that happen, then some world states are impossible conditional on this act and the act can have consequences only in the world states compatible with it. Therefore, the act should be represented by functions into consequences defined only for these latter world states. Of course, these world states should form an event in $\mathscr{A}$ of non-zero probability.

Luce and Krantz call such partial functions conditional decisions and refer the set of the conditional decisions considered with '$\mathscr{D}$'. It is also convenient to follow their convention to fix the domain of such a conditional decision as an subscript on the function symbol, e.g. f_A denotes a function from $A \in \mathscr{A}$ into C.

Accordingly, the expected utility of conditional decisions is calculated by a new formula stated in the subsequent

DEFINITION 4: $\langle \Omega, \mathscr{A}, C, \mathscr{C}, \mathscr{D}; P, V, U \rangle$ is a *special Luce-Krantz-model*[19] iff
(1) $\langle \Omega, \mathscr{A}, P \rangle$ is a σ-additive probability space,
(2) $\langle C, \mathscr{C} \rangle$ is a measurable space,
(3) $\mathscr{D}$ is a non-empty set of $\mathscr{A}$-$\mathscr{C}$-measurable functions into C whose domains are nonnull events in $\mathscr{A}$,
(4) V is a $\mathscr{C}$-measurable function from C into the reals,
(5) for every $f_A \in \mathscr{D}$ the integral $\int_A V \circ f_A \, dP$ exists,
(6) U is the function from $\mathscr{D}$ into the reals for which

$$(6.1) \qquad U(f_A) = \int_A V(f_A(\omega)) \, dP(\omega|A) = \frac{1}{P(A)} \cdot \int_A V \circ f_A \, dP.$$

Now let us have a closer look at this decision model. Is the introduction of conditional decisions really convincing? I think not. The explanation given for them stated that an act should be represented by a function f_A if that act is compatible only with the world states in A while excluding all others. But with which possible world states is an act compatible?

Obviously with only and all those world states in which this act is performed; accordingly the event formed by these world states is just the event of that act being performed. In short, A expresses that f_A is performed. But in special Luce-Krantz-models a probability is assigned to A. Thus we are faced with the unpleasant fact that special Luce–Krantz-models contain hidden probabilities for acts.

This argument can even be weakened. We do not really need to bother whether A expresses that f_A is performed. It suffices to know that $\mathscr{A}$ contains act-dependent events, since acts are supposed to have an influence on world states. But then the model should not contain unconditional probabilities for these events, as was stated at the end of Section 2.

That is, if we want to interpret special Luce–Krantz-models in accordance with the principle stated in Section 2, we have to conceive world states and events as being act-independent. Consequently, there is no point in representing acts as partial functions. Acts can no longer influence world states and have therefore to take into account *all* world states, i.e. acts must be represented by functions defined on the whole of Ω. Thus it seems that we are completely thrown back to Savage's position.

But the situation is even worse. Luce and Krantz make heavy use of conditional decisions in proving their representation theorem. More concretely, to prove their theorem they must, of course, impose rather strong structural assumptions on the set $\mathscr{D}$ of conditional decisions which imply the following: Let $\mathscr{D}_\Omega$ denote the set of all $f_\Omega \in \mathscr{D}$ defined on the whole of Ω. Then $\mathscr{D}$ consists precisely of all restrictions of functions from $\mathscr{D}_\Omega$ to nonnull events in $\mathscr{A}$.[20] That is, to make sense of Luce's and Krantz' metrization we badly need a reasonable interpretation of conditional decisions.

There exists one, though not as genuine acts. For it is still true that a conditional decision f_A $(A \neq \Omega)$ is so to speak an act in a restricted possibility space. As we have seen, it was only necessary to give up the idea that the act itself induces this restriction. How else could it come about? Simply by the experience of the decision maker that A actually obtains. But the decision maker has not, or not yet, had that experience. Hence, the whole affair is to be understood in the following way: Let $f_A \in \mathscr{D}$. If $A = \Omega$, then f_A represents a genuine act. If not, then there is an $f_\Omega \in \mathscr{D}_\Omega$ for which $f_A \subseteq f_\Omega$; this we have just stated. f_Ω represents an act

available to the decision maker; and f_A then represents exactly the *same* act on the hypothesis that the decision maker has learned that A obtains. That is, Luce and Krantz are asking the decision maker not only to preference-order the acts actually open to him, but also to reflect in his preference order how much he would like these acts if he had certain experiences; and here he has to consider all possible experiences.

Luce and Krantz are well aware that there are a great many hypothetical acts among the conditional decisions in $\mathscr{D}$.[21] But they took at least some partial functions as representing actually available acts. In my view, this is not correct; only functions defined on the whole of Ω can stand for available acts, as it is the case in Savage-models.

Indeed, so far we have reached only a reasonable interpretation of special Krantz–Luce-models, but no progress whatsoever beyond Savage. Even the formula (6.1) for expected utilities of conditional decisions offers nothing new, since, having learned that A obtains, a Savagian decision maker would compute the expected utility of an act f_A by (6.1). The only difference between Savage and Luce and Krantz so far lies in their respective metrization procedures, but I will not go into that now.

We have completely neglected the second novelty introduced by Luce and Krantz, though, and this will bring about a genuine generalization of Savage-models, but at a somewhat unexpected place.

7. THE EXPECTED UTILITY FUNCTION IN LUCE'S AND KRANTZ' DECISION MODEL

In special Luce–Krantz-models we included a utility function V for consequences and then defined the expected utility function U by (6.1). This is not what Luce and Krantz do. Similarly to Jeffrey, they operate directly with an expected utility function defined for conditional decisions, which has, of course, to fulfill an averaging condition similar to (5.1). Again, I will state below this condition for the σ-additive case:

DEFINITION 5: $\langle \Omega,\ \mathscr{A},\ C,\ \mathscr{C},\ \mathscr{D},\ P,\ U \rangle$ is a *Luce–Krantz-model* iff
(1) $\langle \Omega,\ \mathscr{A},\ P \rangle$ is a σ-additive probability space,
(2) $\langle C,\ \mathscr{C} \rangle$ is a measurable space,
(3) $\mathscr{D}$ is a non-empty set of $\mathscr{A}$-$\mathscr{C}$-measurable functions into C whose domains are nonnull events in $\mathscr{A}$,

(4) for every $f_A \in \mathcal{D}$ there is an $f_\Omega \in \mathcal{D}$ with $f_A \subseteq f_\Omega$, and if $f_\Omega \in \mathcal{D}$ then every restriction of f_Ω to a nonnull event in $\mathcal{A}$ also is in $\mathcal{D}$,

(5) U is a function from $\mathcal{D}$ into the reals with the following properties:

(a) If $(A_i)_{i \in I}$ is a countable family of mutually disjoint, nonnull events in $\mathcal{A}$, $A = \bigcup_{i \in I} A_i$, $f_{A_i} \in \mathcal{D}$ for all $i \in I$ and $f_A = \bigcup_{i \in I} f_{A_i} \in \mathcal{D}$, then

$$(7.1) \qquad U(f_A) = \sum_{i \in I} U(f_{A_i}) \cdot P(A_i | A),$$

(b) if $f_A, g_B \in \mathcal{D}$ and $f_A(\omega) = g_B(\omega)$ for P-almost all $\omega \in A \cup B$, then

$$(7.2) \qquad U(f_A) = U(g_B).$$

The first part of condition (4) reflects our new interpretation of conditional decisions, whereas the second part of (4) is not necessary for interpretation, but included for technical reasons. As noted earlier, condition (4) is implied by the structural assumptions Luce and Krantz impose on $\mathcal{D}$. (7.1), of course, is the crucial averaging condition, and (7.2) is a rather obvious condition which must be explicitly stated, however, because it does not follow from (7.1).[22]

Now, of course, the question obtrudes with which right the function U of a Luce–Krantz-model may be called an expected utility function, i.e. of what U is the expectation. The remainder of the section will be devoted to this question.

A first hint is that functions of the form (6.1) satisfy (7.1) and (7.2). But definition 5 by no means implies that there is a function V from C into the reals such that U can be represented by (6.1). This indicates that Definition 5 indeed generalizes Definition 4.

As Luce and Krantz have pointed out, however, there are additional assumptions entailing that U is of the form (6.1).[23] To this purpose they have utilized the observation that constant decisions play an essential role in Savage's construction of a utility function V on C from preferences for acts. Since from now on we will make permanent use of constant decisions, it is convenient to introduce the following notation for them: For each $c \in C$ and each nonnull $A \in \mathcal{A}$ we define c_A to be that function on A for which $c_A(\omega) = c$ for all $\omega \in A$. Luce and Krantz have then proved the

THEOREM 1: Let $\langle \Omega, \mathcal{A}, C, \mathcal{C}, \mathcal{D}, P, U \rangle$ be a Krantz–Luce-model.

Suppose that for every $c \in C$ there is a $c_A \in \mathscr{D}$ and that $U(c_A) = U(c_B)$, if $c_A, c_B \in \mathscr{D}$. Then there is a uniquely determined function V from C into the reals such that (6.1) holds for all $f_A \in \mathscr{D}$ with countable range.

Hint of proof: Set $V(c) = U(c_A)$; V is thereby well defined. Next, partition f_A in countably many constant decisions and compute $U(f_A)$ by (7.1) and (7.2).

Theorem 1 tells us which assumptions carry us from Luce–Krantz-models back to special Luce–Krantz-models or to Savage-models. Note also that these assumption can be stated in qualitative terms, as Luce and Krantz actually do.

In a similar fashion the expected utility function of Luce–Krantz-models specializes to that of Jeffrey-models:

THEOREM 2: Let $\langle \Omega, \mathscr{A}, C, \mathscr{C}, \mathscr{D}, P, U \rangle$ be a Luce–Krantz-model. Suppose that $U(f_A) = U(g_A)$ for every $f_A g_A \in \mathscr{D}$. Then there is a uniquely determined function W from $\{A \,|\, P(A) \neq 0\}$ into the reals satisfying (5.1) such that $U(f_A) = W(A)$. Moreover, this function W has the property:

(7.3)   If $P(A) \neq 0$, $P(A \setminus B) = P(B \setminus A) = 0$, then $W(A) = W(B)$.

Again, the assumptions of Theorem 2 can be formulated qualitatively.

The representations of U given in Theorems 1 and 2 are of a rather special kind. One would like to know whether there are more general representations. Now it would be convenient to combine Theorems 1 and 2 additively, i.e. to represent U by a function V according to Theorem 1 and a function W according to Theorem 2 such that for all $f_A \in \mathscr{D}$ with countable range:

(7.4) $$U(f_A) = W(A) + \frac{1}{P(A)} \cdot \int_A V \circ f_A \, dP.$$

As Luce and Krantz themselves have pointed out ([10], p. 262), functions of the form (7.4) satisfy (7.1) and (7.2). But they did not specify conditions implying that U is of the form (7.4). This gap is filled by the next

THEOREM 3: Let $\langle \Omega, \mathscr{A}, C, \mathscr{C}, \mathscr{D}, P, U \rangle$ be a Luce–Krantz-model. Suppose there is a $c^0 \in C$ such that for every $d \in C$ there is an $A \in \mathscr{A}$ with $c_A^0, d_A \in \mathscr{D}$. Suppose further that for all $c_A, c_B, d_A, d_B \in \mathscr{D}$

(7.5) $U(c_A) + U(d_B) = U(c_B) + U(d_A)$.

Finally let there be a function $h_\Omega \in \mathcal{D}_\Omega$ with countable range. Then there is a function V on C and a function W on $\{A \,|\, P(A) \neq 0\}$ fulfilling (5.1) and (7.3) such that (7.4) holds for all $f_A \in \mathcal{D}$ with countable range. Moreover, if the same is true of two other functions V' and W', then there is a real α such that $V'(c) = V(c) + \alpha$ for all $c \in C$ and $W'(A) = W(A) - \alpha$ for all nonnull $A \in \mathcal{A}$.

Proof: Choose $V(c^0)$ arbitrarily. For $d \in C$ define $V(d) = U(d_A) - U(c_A^0) + V(c^0)$ for some A assumed to exist. (7.5) guarantees that V is well defined on C. Define further $W(A) = U(d_A) - V(d)$ for some $d \in C$ if it exists. Again, (7.5) assures that W is well defined on $\mathcal{A}^* = \{A \,|\, d_A \in \mathcal{D}$ for some $d \in C\}$. Also, W satisfies (5.1) and (7.3) on $\mathcal{A}^*$ because of (7.1) and (7.2). Now we extend W onto $\{A \,|\, P(A) \neq 0\}$ in the following way: According to our assumption there is a $h_\Omega \in \mathcal{D}_\Omega$ with countable range $\{c^i \,|\, i \in I\}$. Put $A_i = h_\Omega^{-1}(c^i)$. Then we have $c_{A_i}^i \in \mathcal{D}$ for all nonnull A_i (because of condition (4) of definition 5). Now, take any nonnull $A \in \mathcal{A}$. Then we also have $c_{A_i \cap A}^i \in \mathcal{D}$ for all nonnull $A_i \cap A$. Thus, W is already defined for all nonnull $A_i \cap A$. Hence we can define W for A by applying (7.1) and (7.2). An easy consideration shows that W is thereby well defined for all nonnull events and satisfies (5.1) and (7.3). The uniqueness claim of theorem 3 can be seen to be true by the fact that in the definition of V and W only the value of $V(c^0)$ could be chosen arbitrarily. Finally, take any $f_A \in \mathcal{D}$ with countable range and partition it into countably many constant decisions. A simple computation applying (7.1) and (7.2) then shows that (7.4) holds for f_A.

Theorem 3 has the unpleasant feature that its assumption (7.5) cannot immediately be translated into qualitative terms. The next theorem remedies this defect.

THEOREM 4: Let $\langle \Omega, \mathcal{A}, C, \mathcal{C}, \mathcal{D}, P, U \rangle$ be a Luce–Krantz-model. Suppose there exist $e^1, e^2 \in C$ and disjoint and nonnull $E_1, E_2 \in \mathcal{A}$ such that $e_{E_j}^i \in \mathcal{D}$ $(i, j = 1,2)$, $U(e_{E_1}^1) \neq U(e_{E_1}^2)$, $U(e_{E_1}^i) = U(e_{E_2}^i)$ $(i = 1, 2)$, and $U(e_{E_1}^1 \cup e_{E_2}^2) = U(e_{E_2}^1 \cup e_{E_1}^2)$. If then for $c_A, c_B, d_A, d_B \in \mathcal{D}$ there exist f_{E_1}, f_{E_2}, g_{E_1}, $g_{E_2} \in \mathcal{D}$ such that $U(f_{E_1}) = U(c_A)$, $U(f_{E_2}) = U(c_B)$, $U(g_{E_1}) = U(d_A)$, and $U(g_{E_2}) = U(d_B)$,[24] we have: $U(f_{E_1} \cup g_{E_2}) = U(f_{E_2} \cup g_{E_1})$ iff $U(c_A) + U(d_B) = U(c_B) + U(d_A)$.

Proof: Theorem 4 looks complicated, but is in fact quite trivial. The point of the first assumption is to assure the existence of two disjoint

events E_1 and E_2 such that $P(E_1|E_1 \cup E_2) = P(E_2|E_1 \cup E_2) = \frac{1}{2}$. And then the rest is clear.

It must be added that we can combine (5.2) and (7.4) getting thereby a slight modification of Theorem 3: Under the assumptions of Theorem 3 there exists a real-valued function V_1 on Ω and a real-valued function V_2 on C such that

$$(7.6) \qquad U(f_A) = \frac{1}{P(A)} \cdot \int_A V_1 \, dP + \frac{1}{P(A)} \cdot \int_A V_2 \circ f_A \, dP$$

$$= \frac{1}{P(A)} \cdot \int_A V_1 + V_2 \circ f_A \, dP$$

for every $f_A \in \mathscr{D}$ with countable range. Moreover, if the same holds for V_1' and V_2', then there is a real α such that $V_1' = V_1 - \alpha$ almost everywhere and $V_2' = V_2 + \alpha$.

Have we already reached the most general representation of U as an expectation? Not nearly. For if U is represented additively by (7.4) or (7.6), it still has a rather special form which is implied only by rather subtle assumptions about constant decisions. But we can get along without any such assumptions, as the following theorem shows, which probably yields the most general representation of U.

THEOREM 5: Let $\langle \Omega, \mathscr{A}, C, \mathscr{C}, \mathscr{D}, P, U \rangle$ be a Luce–Krantz-model. Let $\mathscr{D}^*$ be the set of functions which results when we enrich $\mathscr{D}$ by all restrictions of functions from $\mathscr{D}$ to null events. Now suppose there is a possibly uncountable partition $\mathscr{C} \subseteq \mathscr{D}^*$ of $\cup \mathscr{D}^{*25}$ with the following property: For every $f_\Omega \in \mathscr{D}_\Omega$ there is a countable subset $\mathscr{C}_0$ of $\mathscr{C}$ and a null event B from $\mathscr{A}$ such that the restriction of f_Ω to $\Omega \setminus B$ is a subset of $\cup \mathscr{C}_0$. Then there is a function V from $\cup \mathscr{D}^*$ into the reals almost uniquely determined (in a sense to be inferred from the proof) such that for each $f_A \in \mathscr{D}$

$$(7.7) \qquad U(f_A) = \frac{1}{P(A)} \cdot \int_A V(\omega, f_A(\omega)) \, dP(\omega).$$

Proof: Define Q on $\mathscr{D}^*$ by $Q(f_A) = P(A)$ for all $f_A \in \mathscr{D}^*$ and μ on $\mathscr{D}^*$ by

$$\mu(f_A) = \begin{cases} U(f_A) \cdot Q(f_A) & \text{for} \quad f_A \in \mathscr{D} \\ 0 & \text{for} \quad f_A \in \mathscr{D}^* \setminus \mathscr{D} \end{cases}.$$

For $h \in \mathscr{E}$ let further $\mathscr{D}_h^* = \{f_A \in \mathscr{D}^* | f_A \subseteq h\}$, and let Q_h and μ_h, respectively, be the restriction of Q and μ to $\mathscr{D}_h^*$. Then $\mathscr{D}_h^*$ is a σ-algebra over h, Q_h is a finite measure on $\mathscr{D}_h^*$ and, because of (7.1), μ_h is a finite signed measure on $\mathscr{D}_h^*$. Moreover, μ_h is absolutely continuous with respect to Q_h. According to the Radon–Nykodym theorem for signed measures[26] there is a Q_h-almost uniquely determined, $\mathscr{D}_h^*$-measurable function V_h from h into the reals such that for all $f_A \in \mathscr{D}_h \mu_h(f_A) = \int_{f_A} V_h \, dQ_h$.

Finally, we put $V = \bigcup_{h \in \mathscr{E}} V_h$. Since $\mathscr{E}$ is a partition of $\bigcup \mathscr{D}^*$, V is a function from $\bigcup \mathscr{D}^*$ into the reals.

Now, take any f_A from $\mathscr{D}$. Then there is some $f_\Omega \in \mathscr{D}_\Omega$ with $f_A \subseteq f_\Omega$ and, according to our assumption, a countable subset $\mathscr{E}_0$ of $\mathscr{E}$ and a null event B such that the restriction of f_Ω to $\Omega \setminus B$ is a subset of $\bigcup \mathscr{E}_0$. Let A_h be the domain of $f_A \cap h(h \in \mathscr{E}_0)$. For $A' = \bigcup_{h \in \mathscr{E}_0} A_h$ we therefore have $A' \subseteq A$ and $P(A \setminus A') = 0$, and for the restriction $f_{A'}$ of f_A to A' we have $f_{A'} = \bigcup_{h \in \mathscr{E}_0} f_A \cap h$. Hence:

$$\mu(f_A) = \mu(f_{A'}) = \sum_{h \in \mathscr{E}_0} \mu_h(f_A \cap h)$$

$$= \sum_{h \in \mathscr{E}_0} \int_{f_A \cap h} V_h \, dQ_h$$

$$= \sum_{h \in \mathscr{E}_0} \int_{A_h} V(\omega, [f_A \cap h](\omega)) \, dP(\omega)$$

$$= \int_{A'} V(\omega, f_{A'}(\omega)) \, dP(\omega) = \int_A V(\omega, f_A(\omega)) \, dP(\omega).$$

This completes the proof, since $\mu(f_A) = U(f_A) \cdot P(A)$. It remains to note in which sense V is determined almost uniquely: If, for all $h \in \mathscr{E}$, we change the restriction of V to h on some Q_h-null set, then the resulting function satisfies (7.7) too. And every function satisfying (7.7) can be obtained by such a change.

It has also to be mentioned that every expected utility function of the form (7.7) satisfies (7.1) and (7.2).

Let me finally remark that the crucial assumption of Theorem 5 is stated in purely qualitative terms. That is, if we also had some qualitative continuity condition assuring the σ-additivity of the probability measure P and of the expected utility function U (in the sense of satisfying (7.1) in

the denumerable case), Luce's and Krantz' representation theorem and Theorem 5 could be combined to form another representation theorem which would mirror the preference relation among conditional decisions in a numerical order of expressions of the form (7.7).

Theorem 5 probably is in need of some intuitive explanation. As to the crucial premise of Theorem 5, there is nothing much to explain. It is just a rather unintuitive, but technically required structural condition on the set of conditional decisions. Very roughly, it demands that the conditional decisions (in $\mathscr{D}_\Omega$) be not too strongly interlaced with one another. But I think it is a rather weak structural assumption. For example, it is always fulfilled if Ω or $\mathscr{D}_\Omega$ is countable. In fact, it could only be violated if there were uncountably many conditional decisions in $\mathscr{D}_\Omega$ each having uncountably many, non-empty and mutually disjoint intersections with other conditional decisions from $\mathscr{D}_\Omega$. And that is, I think, a rather unlikely situation. It seems to me, however, that this assumption cannot be essentially weakened without losing all hope of generally representing the expected utility function of Luce–Krantz-models as an expectation of something.

Now, how is the function V appearing in Theorem 5 to be understood? It is defined on $\bigcup \mathscr{D}^*$, i.e. on a special subset of $\Omega \times C$. For $\bigcup \mathscr{D}^*$ contains precisely the realizable pairs $\langle \omega, c \rangle \in \Omega \times C$. That is: If $\langle \omega, c \rangle$ is not in $\bigcup \mathscr{D}^*$, then it is impossible, whatever act is chosen, that the world state ω obtains and the consequence c results. Therefore V is a function expressing the utility not of consequences, but of combinations of world states *and* consequences, provided they are jointly realizable.

This is the point where Luce and Krantz generalize Savage's variant of decision theory. They remove Savage's restrictive assumption of utility-independence of world states and consequences.[27] And Theorems 3 and 4 state qualitative conditions yielding this independence, i.e. the additive decomposability of V into two functions V_1 on Ω and V_2 on C such that $V(\omega, c) = V_1(\omega) + V_2(c)$.

Thus, our discussion of Luce–Krantz-models took an unforeseen turn: Luce and Krantz hoped to improve Savage's restrictive representation of the decision maker's subjective probabilities by introducing conditional decisions. Yet conditional decisions failed to do the job they were supposed to do; even in Luce–Krantz-models only functions defined for all world states can be interpreted as representing available acts. But then

we found progress at another place; utilities were more generally represented than in Savage-models.

It must be noted, however, that from a purely quantitative point of view, Luce–Krantz-models are less satisfactory than Fishburn-models. As to utilities, Fishburn-models are equally general as Luce–Krantz-models. But only Fishburn-models overcome the crucial restrictions in the representations of subjective probabilities.

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NOTES

[1] For a survey of this debate cf. Stegmüller [16].

[2] Where we take, for simplicity, preference relations to be observable concepts, though, strictly speaking, they are not.

[3] For a more detailed discussion of these topics see my [15].

[4] Of course, we need not assume that $\mathscr{F}$ is the set of *all* such functions. Savage needed this assumption only for metrization purposes.

[5] This definition obviously is a reformulation of Savage's own definition in usual measure-theoretic terms; there is no essential change except perhaps in assuming σ – instead of finite additivity of P. But we need not bother about that in this context.

[6] He might even assess the acts themselves with absolute, intrinsic utilities besides their expected utilities derived from their consequences. Though this may be a minor point, I find it surprising that almost no one worried about it. In fact, only Jeffrey [7] is able to handle it within his system.

[7] Cf. [5], p. 54, and [7], Section 1.5.

[8] This measure, however, does not remove the restriction stated in note 6 unless one is prepared to represent acts double in Fishburn-models as acts proper and as components of consequences.

[9] Existence and uniqueness of P_f' is assured by the Carathéodory extension theorem. Cf. for example [9], p. 87.

[10] See also [1], p. 58.

[11] I do not mean to imply therewith that it is a wholly absurd thing to attempt to describe decision situations with the help of subjunctive conditionals. Indeed, [6] suggests that such a description might be superior to the standard one. (For a discussion of this claim, see [15].) But I do mean that we should not resort to such obscure things as subjunctive conditionals unless we run into inextricable problems with the standard means.

[12] With the one exception that no utility is assigned to the impossible event.

[13] Taken from [3], p. 337.

[14] Cf. for example [9], p. 132.

[15] I take this objection to be at the basis of Sneed's criticism [14] of Jeffrey, especially of pp. 279f.

[16] For a formulation of a decision model along these lines, where, moreover, propositions are linguistically based according to Carnap [4], see my [15].

[17] This, however, is not the whole truth. In fact, when Savage's small worlds ([12], pp. 82ff) are taken seriously, it seems that a more sophisticated picture has to be made of utilities. For details see my [15].

[18] Strictly speaking, they only assume $\mathscr{A}$ to be an algebra and P to be finitely additive. I am only conforming to my former usage.

[19] I call them 'special' because they still neglect the forementioned second novelty.

[20] Actually, Luce and Krantz [10] require that if f_A, $g_B \in \mathscr{D}$ for disjoint A and B, then $f_A \cup g_B \in \mathscr{D}$, and if $f_A \in \mathscr{D}$, $B \subseteq A$, and $P(B) \neq 0$, then the restriction of f_A to B is in $\mathscr{D}$ (p. 256, axiom 1). Together with the assumption that $\mathscr{D}_\Omega$ is not empty which follows from their axiom 9 (p. 256), this implies the stated property of $\mathscr{D}$.

[21] Cf. [10], pp. 257f and [11].

[22] This is due to the fact that the domain of conditional decisions must be nonnull. In Jeffrey-models where the expected utility function was also defined for non-empty null events such a condition was not needed.

[23] Cf. [10], p. 263, and [8], pp. 391f.

[24] The existence of these latter functions is in fact entailed by axiom 9(i) of Luce and Krantz [10], p. 256.

[25] Here, as in the following, it is essential to conceive functions as set of pairs.

[26] Cf. for example [9], p. 132.

[27] Recall Section 3 of this paper.

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