PHI858 Epistemic Foundations of Game Theory
Final Problem Set

Instructions: Please answer questions 4.8, 4.31 and 9.25 a-e. You should also answer a few additional questions (say, 2-3 questions, the more you answer the higher grade you will receive): You can choose from any of the questions below or the circled questions from the attached problem sets. ***Note that this is only for those of you not writing papers.***

1. Given an extensive game Γ for a finite set of players N where X_i is the set of nodes where player i moves and S_i the set of strategies for player i (functions from X_i to moves for i), and u_i is the utility function for each player. Let $\mathcal{M} = \langle W, \{\sim_i\}_{i \in N}, \mathsf{s}\rangle$ be a model of Γ where $\mathsf{s} : W \to \prod_{i \in N} S_i$. We first fix some notation:

   - $[\![s_i \succ_i t_i]\!]_{\mathcal{M}} = \{w \mid u_i(s_i, \mathsf{s}_{-i}(w)) > u_i(t_i, \mathsf{s}_{-i}(w))\}$ (all the states w where strategy s_i is strictly better than strategy t_i against the strategies of the other players in state w, denoted $\mathsf{s}_{-i}(w)$).
   - $[\![s_i]\!]_{\mathcal{M}} = \{w \mid \mathsf{s}_i(w) = s_i\}$: all states where i is using strategy s_i.
   - Given a strategy profile σ, let $p(\sigma)$ denote the **play** of σ: the sequence of nodes that are reached when all players follow the strategies in σ
   - $[\![x]\!]_{\mathcal{M}} = \{w \mid x \in p(\mathsf{s}(w))\}$: the set of states where vertex x is reached.
   - For a strategy $s_i \in S_i$, let s_i^x denote the strategy in the sub-game starting at node x.
   - $[\![s_i^x \succ_i t_i^x]\!]_{\mathcal{M}} = \{w \mid s_i^x$ is better than t_i^x (yields a strictly higher payoff) against $\mathsf{s}_{-i}(w)\}$ (this is the notion used in the definition of rationality discussed in class)

   We can define three notions of rationality (to simplify notation we define *irrationality*):

   (a) player i is *ex ante irrational*:

   $$\neg R_i^{EA} = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} ([\![s_i]\!]_{\mathcal{M}} \cap K_i[\![t_i \succ_i s_i]\!]_{\mathcal{M}})$$

   all states where i plays a strategy s_i while knowing there is another strategy that is better.

   (b) player i is *reached node irrational*:

   $$\neg R_i^{RN} = \bigcup_{x \in X_i} \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} ([\![x]\!]_{\mathcal{M}} \cap [\![s_i]\!]_{\mathcal{M}} \cap K_i(\neg[\![x]\!]_{\mathcal{M}} \cup [\![t_i \succ_i s_i]\!]_{\mathcal{M}}))$$

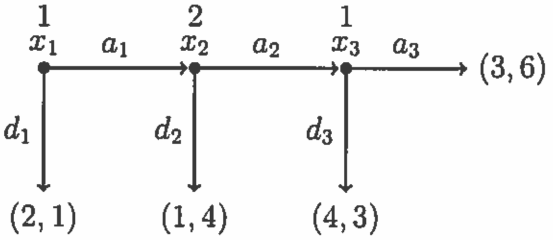
   all states where i plays a strategy s_i while knowing there is another strategy that is better.

(c) player i is *substantively irrational*:

$$\neg R_i^S = \bigcup_{x\in X_i}\bigcup_{s_i\in S_i}\bigcup_{t_i\in S_i}([\![s_i^x]\!]_{\mathcal{M}} \cap K_i([\![t_i^x \succ_i s_i^x]\!]_{\mathcal{M}}))$$

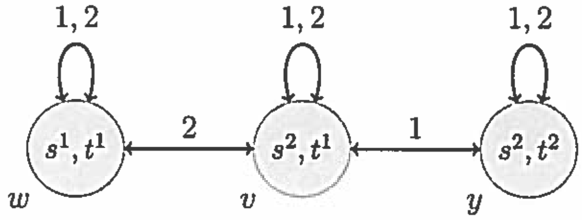
(this is Aumann's notion of rationality discussed in class)

- Consider the following game:



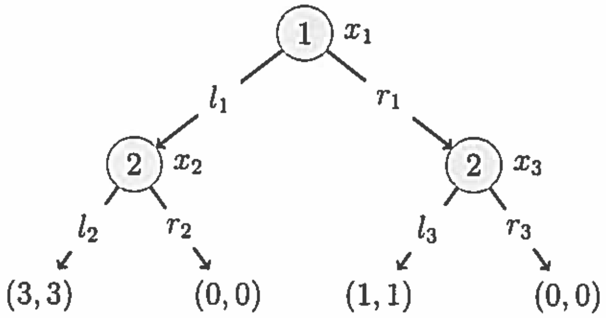
with the four strategies given below and the epistemic model for the above game:

(a) s^1 is the strategy for player 1: $s^1(x_1) = a_1$ and $s^1(x_3) = d_3$
(b) s^2 is the strategy for player 1: $s^2(x_1) = d_1$ and $s^2(x_3) = d_3$
(c) t^1 is the strategy for player 2: $t^1(x_2) = a_2$
(d) t^2 is the strategy for player 2: $t^2(x_2) = d_2$



What are the sets R_2^{EA}, R_2^{RN} and R_2^S? (Give brief explanations for your answers)

- Aumann showed that common knowledge of *substantive rationality* implies that the players will play the BI-solution. Can we weaken the notion of rationality to *ex ante rationality* or *rationality at reached nodes*? ***Hint***: Consider the following game and a model with a single state w assigned the strategy profile (s^1, t^1) where $s^1(x_1) = r_1$ and $t^1(x_2) = r_2$ and $t^1(x_3) = l_3$.



2. Compare and contrast Brian Skyrms' analysis of pure coordination games (e.g., his analysis of the winding road game in Chapter 2 and 3) with one of the analyses discussed in *Explaining Strategic Coordination: Cognitive Hierarchy Theory, Strong Stackelberg Reasoning, and Team Reasoning* by Andrew M. Colman, Briony D. Pulford, and Catherine L. Lawrence.

4.6 Repeat Exercise 4.5, with respect to the following strategic-form game:

Player II

Player I	s_{II}^{1}	s_{II}^{2}	s_{II}^{3}	s_{II}^{4}
s_{I}^{1}	*D*	*D*	*D*	*D*
s_{I}^{2}	I	I	II	II
s_{I}^{3}	II	II	*D*	*D*

4.7 In each of the following games, where Player I is the row player and Player II is the column player, determine whether the process of iterated elimination of strictly dominated strategies yields a single strategy vector when it is completed. If so, what is that vector? Verify that it is the only Nash equilibrium of the game.

	L	*R*
H	4, 2	0, 1
T	1, 1	3, 3

Game A

	L	*R*
H	1, 3	2, 3
T	0, 4	0, 2

Game B

	a	*b*	*c*
γ	1, 0	3, 0	2, 1
β	3, 1	0, 1	1, 2
α	2, 1	1, 6	0, 2

Game C

4.8 What advice would you give to the players in each of the following four games? Provide a detailed justification for all advice given.

Player II

Player I	*L*	*R*
H	2, 3	1, 5
T	0, 0	4, 1

Game A

Player II

Player I	*L*	*R*
H	4, 8	5, 10
T	3, 7	6, 20

Game B

Player II

Player I	*a*	*b*	*c*	*d*
δ	2, 13	4, 8	6, 5	8, 2
γ	6, 4	2, 3	3, 8	8, 4
β	0, 9	7, 7	2, 7	14, 8
α	4, 0	0, 4	4, 6	6, 0

Game C

Player II

Player I	*a*	*b*	*c*	*d*
δ	–5, 5	–4, 4	–5, 1	0, 4
γ	–4, 1	–3, 3	–2, 2	1, 1
β	0, 0	1, –1	–4, 4	–3, 3
α	–1, 6	2, –2	7, 1	5, 2

Game D

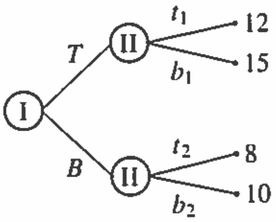
of player j in the game $\widehat{G}$ is greater than or equal to his maxmin value in G. Is the maxmin value of player i in game $\widehat{G}$ necessarily less than his maxmin value in G? Prove this last statement, or find a counterexample.

4.28 Find an example of a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ in strategic form such that the game $\widehat{G}$ derived from G by elimination of one strategy in one player's strategy set has an equilibrium that is not an equilibrium in the game G.

4.29 Prove Corollary 4.33 on page 108: let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic form game and let $\widehat{G}$ be the game derived from G by iterative elimination of dominated strategies. Then every equilibrium s^* in the game $\widehat{G}$ is also an equilibrium in the game G.

4.30 Find an example of a strategic form game G and of an equilibrium s^* of that game such that for each player $i \in N$ the strategy s_i^* is dominated.

4.31 The following questions relate to the following two-player zero-sum game.



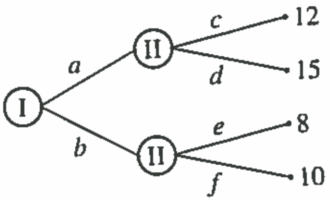
(a) Find an optimal strategy for each player by applying backward induction.
(b) Describe this game in strategic form.
(c) Find all the optimal strategies of the two players.
(d) Explain why there are optimal strategies in addition to the one you identified b backward induction.

4.32 (a) Let $A = (a_{ij})$ be an $n \times m$ matrix representing a two-player zero-sum gam where the row player is Ann and the column player is Bill. Let $B = (b_{ji})$ be new $m \times n$ matrix in which the row player is Bill and the column player is An What is the relation between the matrices A and B?

(b) Conduct a similar transformation of the names of the players in the followir matrix and write down the new matrix.

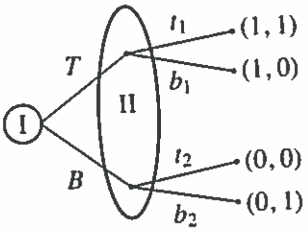
		Player II		
		L	*M*	*R*
Player I	*T*	3	−5	7
	B	−2	8	4

7.3 Find all the equilibria of the following two-player zero-sum game.

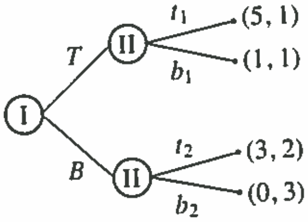


Explain why one cannot obtain all the equilibria of the game by implementing backward induction.

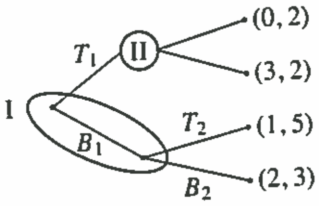
7.4 Find all the subgame perfect equilibria of the following games.



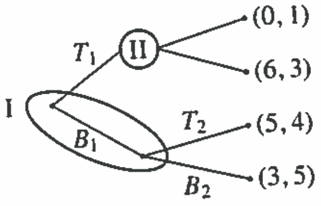
Game A



Game B



Game C



Game D

7.5 **The Ultimatum game** Allen and Rick need to divide \$100 between them as follows: first Allen suggests an integer x between 0 and 100 (which is the amount of money he wants for himself). Rick, on hearing the suggested amount, decides whether to accept or reject. If Rick accepts, the payoff of the game is $(x, 1 - x)$: Allen receives x dollars, and Rick receives $100 - x$ dollars. If Rick chooses to reject, neither player receives any money.

(a) Describe this situation as an extensive-form game.
(b) What is the set of pure strategies each player has?
(c) Show that any result $(a, 100 - a)$, $a \in \{0, 1, \dots, 100\}$, is a Nash equilibrium payoff. What are the corresponding equilibrium strategies?
(d) Find all the subgame perfect equilibria of this game.

7.6 **The Judgment of Solomon** Elizabeth and Mary appear before King Solomon at his palace, along with an infant. Each woman claims that the infant is her child. The

[1953]. Exercise 9.25 is proved in Geanakoplos and Polemarchakis [1982], from which Exercise 9.28 is also taken. Exercise 9.26 was donated to the authors by Ayala Mashiah-Yaakovi. Exercises 9.29 and 9.30 are from Geanakoplos and Sebenius [1983]. Exercise 9.33 is taken from Geanakoplos [1992]. Exercise 9.34 is the famous "coordinated attack problem," studied in the field of distributed computing. The formulation of the exercise is from Halpern [1986]. Exercise 9.39 is from Harsanyi [1968a]. Exercise 9.40 is based on Spence [1974]. Exercise 9.41 is based on Akerlof [1970]. Exercise 9.46 is the "Electronic Mail game" of Rubinstein [1989]. Exercise 9.53 is based on Aumann [1987].

The authors thank Yaron Azrieli, Aviad Heifetz, Dov Samet, and Eran Shmaya for their comments on this chapter.

9.8 Exercises

In the exercises in this chapter, all announcements made by the players are considered common knowledge, and the game of each exercise is also considered common knowledge among the players.

9.1 Prove that the knowledge operator K_i (Definition 9.8, page 325) of each player i satisfies the following properties:

(a) $K_i Y = Y$: player i knows that Y is the set of all states.
(b) $K_i A \cap K_i B = K_i(A \cap B)$: player i knows event A and knows event B if and only if he knows event $A \cap B$.
(c) $(K_i A)^c = K_i((K_i A)^c)$: player i does not know event A if and only if he knows that he does not know event A.

9.2 This exercise shows that the Kripke S5 system characterizes the knowledge operator.

Let Y be a finite set, and let $K : 2^Y \to 2^Y$ be an operator that associates with each subset A of Y a subset $K(A)$ of Y. Suppose that the operator K satisfies the following properties:

(i) $K(Y) = Y$.
(ii) $K(A) \cap K(B) = K(A \cap B)$ for every pair of subsets $A, B \subseteq Y$.
(iii) $K(A) \subseteq A$ for every subset $A \subseteq Y$.
(iv) $K(K(A)) = K(A)$ for every subset $A \subseteq Y$.
(v) $(K(A))^c = K((K(A))^c)$ for every subset $A \subseteq Y$.

Associate with each $\omega \in Y$ a set $F(\omega)$ as follows:

$$F(\omega) := \bigcap \{A \subseteq Y, \omega \in K(A)\}. \tag{9.118}$$

(a) Prove that $\omega \in F(\omega)$ for each $\omega \in Y$.
(b) Prove that if $\omega' \in F(\omega)$, then $F(\omega) = F(\omega')$. Conclude from this that the family of sets $\mathcal{F} := \{F(\omega), \omega \in Y\}$ is a partition of Y.

(c) Let K' be the knowledge operator defined by the partition $\mathcal{F}$:

$$K'(A) = \{\omega \in Y : F(\omega) \subseteq A\}. \tag{9.119}$$

Prove that $K' = K$.

(d) Which of the five properties listed above did you use in order to prove that $K' = K$?

9.3 Prove that in Kripke's S5 system (see page 327), the fourth property, $K_i K_i A = K_i A$, is a consequence of the other four properties.

9.4 Consider an Aumann model of incomplete information in which

$N = \{1, 2\}$,
$Y = \{1, 2, 3, 4, 5, 6, 7\}$,
$\mathcal{F}_1 = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$,
$\mathcal{F}_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}\}$.

Let $A = \{1\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Identify the events K_1A, K_2A, K_2K_1A, K_1K_2A, K_1B, K_2B, K_2K_1B, K_1K_2B, $K_1K_2K_1B$, $K_2K_1K_2B$.

9.5 Emily, Marc, and Thomas meet at a party to which novelists and poets have been invited. Every attendee at the party is either a novelist or a poet (but not both). Every poet knows all the other poets, but every novelist does not know any of the other attendees, whether they are poets or novelists. What do Emily, Marc, and Thomas know about each other's professions? Provide an Aumann model of incomplete information that describes this situation (there are several ways to do so).

9.6 I love Juliet, and I know that Juliet loves me, but I do not know if Juliet knows that I love her. Provide an Aumann model of incomplete information that describes this situation, and specify a state of the world in that model that corresponds to this situation (there are several possible ways of including higher-order beliefs in this model).

9.7 Construct an Aumann model of incomplete information for each of the following situations, and specify a state of the world in that model which corresponds to the situation (there are several possible ways of including higher-order beliefs in each model):

(a) Mary gave birth to a baby, and Herod knows it.
(b) Mary gave birth to a baby, and Herod does not know it.
(c) Mary gave birth to a baby, Herod knows it, and Mary knows that Herod knows it.
(d) Mary gave birth to a baby, Herod knows it, but Mary does not know that Herod knows it.
(e) Mary gave birth to a baby, Herod does not know it, and Mary does not know whether Herod knows it or not.

9.8 Romeo composes a letter to Juliet, and gives it to Tybalt to deliver to Juliet. While on the way, Tybalt peeks at the letter's contents. Tybalt gives Juliet the letter, and

(c) A stranger enters the room, holding a bell. Once a minute, he rings the bell while saying "If you know that the color of the hat on your head is red, leave this room immediately." Does anyone leave the room after a few rings? Why?

(d) At a certain point in time, the announcer says, "At least one of you is wearing a red hat." He continues to ring the bell once a minute and requesting that those who know their hat to be red to leave. Use the Aumann model of incomplete information to prove that after the third ring, all three hat-wearers will leave the room.

(e) What information did the announcer add by saying that at least one person in the room was wearing a red hat, when this was known to everyone before the announcement was made?
Hint: See Example 9.12 on page 327.

(f) Generalize this result to n individuals (instead of 3).

9.24 Prove that in an Aumann model of incomplete information with a common prior **P**, if in a state of the world ω Player 1 knows that Player 2 knows A, then $\mathbf{P}(A \mid F_1(\omega)) = 1$.

9.25 Consider an Aumann model of incomplete information with beliefs in which

$N = \{\text{I}, \text{II}\}$,
$Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
$\mathcal{F}_{\text{I}} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$,
$\mathcal{F}_{\text{II}} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9\}\}$,
$\mathbf{P}(\omega) = \frac{1}{9}, \quad \forall \omega \in Y$.

Let $A = \{1, 5, 9\}$, and suppose that the true state of the world is $\omega_* = 9$. Answer the following questions:

(a) What is the probability that Player I (given his information) ascribes to the event A?

(b) What is the probability that Player II ascribes to the event A?

(c) Suppose that Player I announces the probability you calculated in item (a) above. How will that affect the probability that Player II now ascribes to the event A?

(d) Suppose that Player II announces the probability you calculated in item (c). How will that affect the probability that Player I ascribes to the event A, after hearing Player II's announcement?

(e) Repeat the previous two questions, with each player updating his conditional probability following the announcement of the other player. What is the sequence of conditional probabilities the players calculate? Does the sequence converge, or oscillate periodically (or neither)?

(f) Repeat the above, with $\omega_* = 8$.

(g) Repeat the above, with $\omega_* = 6$.

(h) Repeat the above, with $\omega_* = 4$.

(i) Repeat the above, with $\omega_* = 1$.

(a) Neither Nicolas nor Marc knows the company's true value; both ascribe probability $\frac{1}{3}$ to each possible value.

(b) Nicolas knows the company's true value, whereas Marc does not know it, and ascribes probability $\frac{1}{3}$ to each possible value.

(c) Marc does not know the company's worth and ascribes probability $\frac{1}{3}$ to each possible value. Marc further ascribes probability p to the event that Nicolas knows the value of the company, and probability $1 - p$ to the event that Nicolas does not know the value of the company, and instead ascribes probability $\frac{1}{3}$ to each possible value.

9.43 Prove that in each game with incomplete information with a finite set of players, where the set of types of each player is a countable set, and the set of possible actions of each type is finite, there exists a Bayesian equilibrium (in behavior strategies).

Guidance: Suppose that the set of types of player i, T_i, is the set of natural numbers $\mathbb{N}$. Denote $T_i^k := \{1, 2, \ldots, k\}$ and $T^k = \times_{i \in N} T_i^k$. Let p^k be the probability distribution p conditioned on the set T^k:

$$p^k(t) = \begin{cases} \frac{p(t)}{p(T^k)} & t \in T^k, \\ 0 & t \notin T^k. \end{cases} \tag{9.125}$$

Prove that for a sufficiently large k, the denominator $p(T^k)$ is positive and therefore the probability distribution p^k is well defined. Show that for each k, the game in which the probability distribution over the types is p^k has an equilibrium, and any accumulation point of such equilibria, as k goes to infinity, is an equilibrium of the original game.

9.44 Prove Theorem 9.51 on page 354: a strategy vector $\sigma^* = (\sigma_i^*)_{i \in N}$ is a Bayesian equilibrium in a game Γ with incomplete information if and only if the strategy vector $(\sigma_i^*(t_i))_{i \in N, t_i \in T_i}$ is a Nash equilibrium in the agent-form game $\widehat{\Gamma}$. (For the definition of an agent-form game, see Definition 9.50 on page 354.)

9.45 This exercise shows that in a game with incomplete information, the payoff function of an inactive type has no effect on the set of equilibria. Let $\Gamma = (N, (T_i)_{i \in N}, p, S, (s_t)_{t \in \times_{i \in N} T_i})$, where $s_t = (N, (A_i(t_i), u_i(t))_{i \in N})$ for each $t \in \times_{i \in N} T_i$, be a game with incomplete information in which there exists a player j and a type t_j^* of player j such that $|A_j(t_j^*)| = 1$. Let $\widehat{\Gamma}$ be a game with incomplete information that is identical to Γ, except that the payoff function $\widehat{u}_j(t_j^*)$ of player j of type t_j^* may be different from $u_j(t_j^*)$, that is, $\widehat{u}_i(t; a) = u_i(t; a)$ if $t_j \neq t_j^*$ or $i \neq j$. Show that the two games Γ and $\widehat{\Gamma}$ have the same set of Bayesian equilibria.

9.46 **Electronic Mail game** Let $L > M > 0$ be two positive real numbers. Two players play a game in which the payoff function is one of the following two, depending on the value of the state of nature s, which may be 1 or 2:

		Player II	
		A	B
Player I	A	M, M	$1, -L$
	B	$-L, 0$	$0, 0$

The state game for $s = 1$

		Player II	
		A	B
Player I	A	$0, 0$	$0, -L$
	B	$-L, 1$	M, M

The state game for $s = 2$

The probability that the state of nature is $s = 2$ is $p < \frac{1}{2}$. Player I knows the true state of nature, and Player II does not know it. The players would clearly prefer to coordinate their actions and play (A, A) if the state of nature is $s = 1$ and (B, B) if the state is $s = 2$, which requires that both of them know what the true state is. Suppose the players are on opposite sides of the globe, and the sole method of communication available to them is e-mail. Due to possible technical communications disruptions, there is a probability of $\varepsilon > 0$ that any e-mail message will fail to arrive at its destination. In order to transfer information regarding the state of nature from Player I to Player II, the two players have constructed an automated system that sends e-mail from Player I to Player II if the state of nature is $s = 2$, and does not send any e-mail if the state is $s = 1$. To ensure that Player I knows that Player II received the message, the system also sends an automated confirmation of receipt of the message (by e-mail, of course) from Player II to Player I the instant Player I's message arrives at Player II's e-mail inbox. To ensure that Player II knows that Player I received the confirmation message, the system also sends an automated confirmation of receipt of the confirmation message from Player I to Player II the instant Player II's confirmation arrives at Player I's e-mail inbox. The system then proceeds to send an automated confirmation of the receipt of the confirmation of the receipt of the confirmation, and so forth. If any of these e-mail messages fail to arrive at their destinations, the automated system stops sending new messages. After communication between the players is completed, each player is called upon to choose an action, A or B.

Answer the following questions:

(a) Depict the situation as a game with incomplete information, in which each type of each player is indexed by the number of e-mail messages he has received.

(b) Prove that the unique Bayesian equilibrium where Player I plays A when $s = 1$ is for both players to play A under all conditions.

(c) How would you play if you received 100 e-mail confirmation messages? Explain your answer.

9.47 In the example described in Section 9.5 (page 361), for each $\varepsilon \in [0, 1]$ find Bayesian equilibria in threshold strategies, where α has uniform distribution over the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$ and β has uniform distribution over the interval $\left[-\frac{1}{3}, \frac{2}{3}\right]$.

9.48 In each of the two strategic-form games whose matrices appear below, find all the equilibria. For each equilibrium, describe a sequence of games with incomplete