9 Games with incomplete information and common priors

Chapter summary

In this chapter we study situations in which players do not have complete information on the environment they face. Due to the interactive nature of the game, modeling such situations involves not only the *knowledge* and *beliefs* of the players, but also the whole *hierarchy of knowledge* of each player, that is, knowledge of the knowledge of the other players, knowledge of the knowledge of the other players of the knowledge of other players, and so on. When the players have beliefs (i.e. probability distributions) on the unknown parameters that define the game, we similarly run into the need to consider *infinite hierarchies of beliefs*. The challenge of the theory was to incorporate these infinite hierarchies of knowledge and beliefs in a workable model.

We start by presenting the Aumann model of incomplete information, which models the knowledge of the players regarding the payoff-relevant parameters in the situation that they face. We define the *knowledge operator*, the concept of *common knowledge*, and characterize the collection of events that are common knowledge among the players.

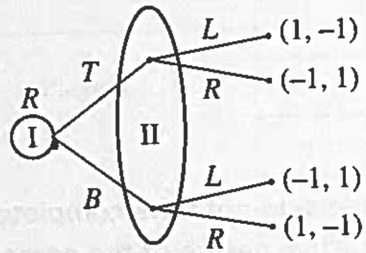
We then add to the model the notion of belief and prove Aumann's agreement theorem: it cannot be common knowledge among the players that they disagree about the probability of a certain event.

An equivalent model to the Aumann model of incomplete information is a *Harsanyi game with incomplete information*. After presenting the game, we define two notions of equilibrium: the Nash equilibrium corresponding to the *ex ante* stage, before players receive information on the game they face, and the Bayesian equilibrium corresponding to the *interim* stage, after the players have received information. We prove that in a Harsanyi game these two concepts are equivalent.

Finally, using games with incomplete information, we present Harsanyi's interpretation of mixed strategies.

As we have seen, a very large number of real-life situations can be modeled and analyzed using extensive-form and strategic-form games. Yet, as Example 9.1 shows, there are situations that cannot be modeled using those tools alone.

Example 9.1 Consider the Matching Pennies game, which is depicted in Figure 9.1 in both extensive form and strategic form.



		Player II: L	Player II: R
Player I	T	1, −1	−1, 1
Player I	B	−1, 1	1, −1

Figure 9.1 The game of Matching Pennies, in extensive form and strategic form

Suppose that Player I knows that he is playing Matching Pennies, but believes that Player II does not know that the pure strategy R is available to her. In other words, Player I believes that Player II is convinced that she has only one pure strategy, L. Suppose further that Player II does in fact know that she (Player II) is playing Matching Pennies, with both pure strategies available. How can we model this game? Neither the extensive-form nor the strategic-form descriptions of the game enable us to model such a state of players' knowledge and beliefs. If we try to analyze this situation using only the depictions of the game appearing in Figure 9.1, we will not be able to predict how the players will play, or recommend an optimal course of action.

For example, as we showed on page 52, the optimal strategy of Player I playing Matching Pennies is the mixed strategy $[\frac{1}{2}(T), \frac{1}{2}(B)]$. But in the situation we have just described, Player I believes that Player II will play L, so that his best reply is the pure strategy T.

Note that Player I's optimal strategy depends only on how he perceives the game: what he knows about the game and what he believes Player II knows about the game. The way that Player II really perceives the game (which is not necessarily known to Player I) has no effect on the strategy chosen by Player I.

Consider next a slightly more complicated situation, in which Player I knows that he is playing Matching Pennies, he believes that Player II knows that she is playing Matching Pennies, and he believes that Player II believes that Player I does not know that the pure strategy B is available to him. Then Player I will believe that Player II believes that Player I will play strategy T, and he will therefore conclude that Player II will select strategy R, and Player I's best strategy will therefore be B.

A similar situation obtains if there is incomplete information regarding some of the payoffs. For example, suppose that Player I knows that his payoff under the strategy profile (T, L) is 5 rather than 1, but believes that Player II does not know this, and that she thinks the payoff is 1. How should Player I play in this situation? Or consider an even more complicated situation, in which both Player I and Player II know that Player I's payoff under (T, L) is 5, but Player II believes Player I does not know that she (Player II) knows this; Player II believes Player I believes Player II thinks the payoff is 1. ◄

Situations like those described in Example 9.1, in which players do not necessarily know which game is being played, or are uncertain about whether the other players know which game is being played, or are uncertain whether the other players know whether the other players know which game is being played, and so on, are called situations of "incomplete information." In this chapter we study such situations, and see how they can be modeled and analyzed as games.

Notice that neither of the situations described in Example 9.1 is well defined, as we have not precisely defined what the players know. For example, in the second case we did not specify what Player I knows about what Player II knows about what Player I knows about what Player II knows, and we did not touch upon what Player II knows. Consideration of hierarchies of levels of knowledge leads to the concept of common knowledge, which we touched upon in Section 4.5 (page 87). An informal definition of common knowledge is:

Definition 9.2 *A fact F is* common knowledge *among the players of a game if all the players know F, all the players know that all the players know F, all the players know that all the players know that all the players know F, and so on (for every finite number of levels).*[1]

Definition 9.2 is incomplete, because we have not yet defined what we mean by a "fact," nor have we defined the significance of the expression "knowing a fact." These concepts will be modeled formally later in this chapter, but for now we will continue with an informal exposition.

So far we have seen that in situations involving several players, incomplete knowledge of the game that is being played leads us to consider infinite hierarchies of knowledge. In decision-making situations with incomplete information, describing the information that decision makers have usually cannot be captured by labeling a given fact as "known" or "unknown." Decision makers often have assessments or beliefs about the truthfulness of various facts. For example, when a person takes out a variable-rate loan he never has precise knowledge of the future fluctuations of the interest rate (which can significantly affect the total amount of loan repayment), but he may have certain beliefs about future rates, such as "I assign probability 0.7 to the event that there will be lower interest rates over the term of the loan." To take another example, a company bidding for oil exploration rights in a certain geographical location has beliefs about the amount of oil likely to be found there and the depth of drilling required (which affects costs and therefore expected profits). A trial jury passing judgment on a defendant expresses certain collective beliefs about the question: is the defendant guilty as charged? For our purposes in this chapter, the source of such probabilistic assessments is of no importance. The assessments may be based on "objective" measurements such as geological surveys (as in the oil exploration example), on impressions (as in the case of a jury deliberating the judgment it will render in a trial), or on personal hunches and information published in the media (as in the example of the variable-rate loan). Thus, probability assessments may be objective or subjective.[2] In our models, a decision maker's beliefs will be expressed by a probability distribution function over the possible values of parameters unknown to him.

1 A simple example of a fact that is common knowledge is a *public* event: when a teacher is standing before a class, that fact is common knowledge among the students, because every student knows that every student knows ... that the teacher is standing before the class.

2 A formal model for deriving an individual's subjective probability from his preferences was first put forward by Savage [1954], and later by Anscombe and Aumann [1963] (see also Section 2.8 on page 26).

The most widely accepted statistical approach for dealing with decision problems in situations of incomplete information is the *Bayesian approach*.[3] In the Bayesian approach, every decision maker has a probability distribution over parameters that are unknown to him, and he chooses his actions based on his beliefs as expressed by that distribution. When several decision makers (or players) interact, knowing the probability distribution (beliefs) of each individual decision maker is insufficient: we also need to know what each one's beliefs are about the beliefs of the other decision makers, what they believe about his beliefs about the others' beliefs, and so on. This point is illustrated by the following example.

Example 9.1 *(Continued)* Returning to the Matching Pennies example, suppose that Player I attributes probability p_I to the event: "Player II knows that R is a possible action." The action that Player I will choose clearly depends on p_I, because the entire situation hinges on the value of p_I: if $p_I = 1$, Player I believes that Player II knows that R is an action available to her, and if $p_I = 0$, he believes that Player II does not know that R is possible at all. If $0 < p_I < 1$, Player I believes that it is possible that Player II knows that R is an available strategy. But the action chosen by Player I also depends on his beliefs about the beliefs of Player II: because Player I's action depends on p_I, it follows that Player II's action depends on her beliefs about p_I, namely, on her beliefs about Player I's beliefs. By the same reasoning, Player I's action depends on his beliefs about Player II's beliefs about his own beliefs, p_I. As in the case of hierarchy of knowledge, we see that determining the best course of action of a Player requires considering an infinite hierarchy of beliefs. ◄

Adding beliefs to our model is a natural step, but it leads us to an infinite hierarchy of beliefs. The concepts of knowledge and of beliefs are closely intertwined in games of incomplete information. For didactic reasons, however, we will treat the two notions separately, considering first hierarchies of knowledge and then hierarchies of beliefs.

9.1 The Aumann model of incomplete information and the concept of knowledge

In this section we will provide a formal definition of the concept of "knowledge," and then construct hierarchies of knowledge: what each player knows about what the other players know. We will start with an example to illustrate the basic elements of the model.

Example 9.3 Assume that racing cars are produced in three possible colors: gold, red, and purple. Color-blind individuals cannot distinguish between red and gold. Everyone knows that John is color-blind, but no one except Paul knows whether or not Paul is color-blind too. John and Paul are standing side by side viewing a photograph of the racing car that has just won first prize in the Grand Prix, and asking themselves what color it is. The parameter that is of interest in this example is the color of

3 The Bayesian approach is named after Thomas Bayes, 1702–1761, a British clergyman and mathematician who formulated a special case of the rule now known as Bayes' rule.

the car, which will later be called the *state of nature*, and we wish to describe the knowledge that the players possess regarding this parameter.

If the color of the car is purple, then both color-blind and non-color-blind individuals know that fact, so that both John and Paul know that the car is purple, and each of them knows that the other knows that the car is purple. If, however, the car is red or gold, then John knows that it is either red or gold. As he does not know whether or not Paul is color-blind, he does not know whether Paul knows the exact color of the car. Because Paul knows that John is color-blind, if the car is red or gold he knows that John does not know what the precise color is, and John knows that Paul knows this.

We therefore need to consider six distinct possibilities (three possibilities per car color times two possibilities regarding whether or not Paul is color-blind):

- The car is purple and Paul is not color-blind. John and Paul both know that the car is purple, they each know that the other knows that the car is purple, and so on.
- The car is purple and Paul is color-blind. Here, too, John and Paul both know that the car is purple, they each know that the other knows that the car is purple, and so on.
- The car is red and Paul is not color-blind. Paul knows the car is red; John knows that the car is red or gold; John does not know whether or not Paul knows the color of the car.
- The car is gold and Paul is not color-blind. Paul knows the car is gold; John knows that the car is red or gold; John does not know whether or not Paul knows the color of the car.
- The car is red and Paul is color-blind. Paul and John know that the car is red or gold; John does not know whether or not Paul knows the color of the car.
- The car is gold and Paul is color-blind. Paul and John know that the car is red or gold; John does not know whether or not Paul knows the color of the car.

In each of these possibilities, both John and Paul clearly know more than we have explicitly written above. For example, in the latter four situations, Paul knows that John does not know whether Paul knows the color of the car. Each of the six cases is associated with what will be defined below as a *state of the world*, which is a description of a state of nature (in this case, the color of the car) and the state of knowledge of the players. Note that the first two cases describe the same state of the world, because the difference between them (Paul's color-blindness) affects neither the color of the car, which is the parameter that is of interest to us, nor the knowledge of the players regarding the color of the car. ◀

The definition of the set of states of nature depends on the situation that we are analyzing. In Example 9.3 the color of the car was the focus of our interest – perhaps, for example, because a bet has been made regarding the color. Since the most relevant parameters in a game are the payoffs, in general we will want the states of nature to describe all the parameters that affect the payoffs of the players (these are therefore also called "payoff-relevant parameters"). For instance, if in Example 9.3 we were in a situation in which Paul's color-blindness (or lack thereof) were to affect his utility, then color-blindness would be a payoff-relevant parameter and would comprise a part of the description of the state of nature. In such a model there would be six distinct states of nature, rather than three.

Definition 9.4 *Let S be a finite set of states of nature. An* Aumann model of incomplete information *(over the set S of states of nature) consists of four components $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathbf{s})$, where:*

- *N is a finite set of players;*
- *Y is a finite set of elements called* states of the world;[4]
- *$\mathcal{F}_i$ is a partition of Y, for each $i \in N$ (i.e., a collection of disjoint nonempty subsets of Y whose union is Y);*
- *$\mathfrak{s} : Y \to S$ is a function associating each state of the world with a state of nature.*

The interpretation is that if the "true" state of the world is ω_*, then each player $i \in N$ knows only the element of his partition $\mathcal{F}_i$ that contains ω_*. For example, if $Y = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F}_i = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, then player i cannot distinguish between ω_1 and ω_2. In other words, if the state of the world is ω_1, player i knows that the state of the world is either ω_1 or ω_2, and therefore knows that the state of the world is not ω_3. For this reason, the partition $\mathcal{F}_i$ is also called the *information* of player i. The element of the partition $\mathcal{F}_i$ that contains the state of the world ω is denoted $F_i(\omega)$. For convenience, we will use the expression "the information of player i" to refer both to the partition $\mathcal{F}_i$ and to the partition element $F_i(\omega_*)$ containing the true state of the world.

Definition 9.5 *An* Aumann situation of incomplete information *over a set of states of nature S is a quintuple $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \omega_*)$, where $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ is an Aumann model of incomplete information and $\omega_* \in Y$.*

The state ω_* is the "true state of the world" and each player knows the partition element $F_i(\omega_*)$ in his information partition that contains the true state. A situation of incomplete information describes a knowledge structure at a particular state of the world, i.e., in a particular reality. Models of incomplete information, in contrast, enable us to analyze all possible situations.

Example 9.3 *(Continued)* An Aumann model of incomplete information for this example is as follows:

- $N = \{\text{John, Paul}\}$.
- $S = \{\text{Purple Car, Red Car, Gold Car}\}$.
- $Y = \{\omega_{g,1}, \omega_{r,1}, \omega_{g,2}, \omega_{r,2}, \omega_p\}$.
- John's partition is $\mathcal{F}_J = \{\{\omega_{g,1}, \omega_{g,2}, \omega_{r,1}, \omega_{r,2}\}, \{\omega_p\}\}$.
- Paul's partition is $\mathcal{F}_P = \{\{\omega_{g,1}, \omega_{r,1}\}, \{\omega_{g,2}\}, \{\omega_{r,2}\}, \{\omega_p\}\}$.
- The function $\mathfrak{s}$ is defined by

$$\mathfrak{s}(\omega_{g,1}) = \mathfrak{s}(\omega_{g,2}) = \text{Gold Car}, \mathfrak{s}(\omega_{r,1}) = \mathfrak{s}(\omega_{r,2}) = \text{Red Car}, \qquad \mathfrak{s}(\omega_p) = \text{Purple Car}.$$

The state of the world ω_p is associated with the situation in which the car is purple, in which case both John and Paul know that it is purple, and each of them knows that the other knows that the car is purple. It represents the two situations in the two first bullets on page 323, which differ only in whether Paul is color-blind or not. As we said before, these two situations are equivalent, and can be represented by the same state of the world, as long as Paul's color-blindness is not

4 We will later examine the case where Y is infinite, and show that some of the results obtained in this chapter also hold in that case.

payoff relevant, and hence is not part of the description of the state of nature. The state of the world $\omega_{g,1}$ is associated with the situation in which the car is gold and Paul is color-blind, while the state of the world $\omega_{r,1}$ is associated with the situation in which the car is red and Paul is color-blind; in both these situations, Paul cannot distinguish which state of the world holds, because he is color-blind and cannot tell red from gold. The state of the world $\omega_{g,2}$ is associated with the situation in which the car is gold and Paul is not color-blind, while the state of the world $\omega_{r,2}$ is associated with the situation in which the car is red and Paul is not color-blind; in both these cases Paul knows the true color of the car. Therefore, $F_P(\omega_{g,2}) = \{\omega_{g,2}\}$, and $F_P(\omega_{g,1}) = \{\omega_{g,1}, \omega_{r,1}\}$.

As for John, he is both color-blind and does not know whether Paul is color-blind. He therefore cannot distinguish between the four states of the world $\{\omega_{g,1}, \omega_{r,1}, \omega_{g,2}, \omega_{r,2}\}$, so that $F_J(\omega_{g,1}) = F_J(\omega_{g,2}) = F_J(\omega_{r,1}) = F_J(\omega_{r,2}) = \{\omega_{g,1}, \omega_{r,1}, \omega_{g,2}, \omega_{r,2}\}$.

The true state of the world is one of the possible states in the set Y. The Aumann model along with the true state of the world describes the actual situation faced by John and Paul. ◀

Definition 9.6 *An* event *is a subset of* Y.

In Example 9.3 the event $\{\omega_{g,1}, \omega_{g,2}\}$ is the formal expression of the sentence "the car is gold," while the event $\{\omega_{g,1}, \omega_{g,2}, \omega_p\}$ is the formal expression of the sentence "the car is either gold or purple."

We say that an event A *obtains* in a state of the world ω if $\omega \in A$. It follows that if event A obtains in a state of the world ω and if $A \subseteq B$, then event B obtains in ω.

Definition 9.7 *Let* $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ *be an Aumann model of incomplete information, let* i *be a player, let* $\omega \in Y$ *be a state of the world, and let* $A \subseteq Y$ *be an event.* Player i knows A in ω *if*

$$F_i(\omega) \subseteq A. \tag{9.1}$$

If $F_i(\omega) \subseteq A$, then in state of the world ω player i knows that event A obtains (even though he may not know that the state of the world is ω), because according to his information, all the possible states of the world, $F_i(\omega)$, are included in the event A.

Definition 9.8 *Let* $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ *be an Aumann model of incomplete information, let* i *be a player, and let* $A \subseteq Y$ *be an event. Define an operator* $K_i : 2^Y \to 2^Y$ *by*[5]

$$K_i(A) := \{\omega \in Y : F_i(\omega) \subseteq A\}. \tag{9.2}$$

We will often denote $K_i(A)$, the set of all states of the world in which player i knows event A, by K_iA. Thus, player i knows event A in state of the world ω_* if and only if $\omega_* \in K_iA$. The definition implies that the set K_iA equals the union of all the elements in the partition $\mathcal{F}_i$ contained in A. The event $K_j(K_iA)$ (which we will write as K_jK_iA for short) is the event that player j knows that player i knows A:

$$K_jK_iA = \{\omega \in Y : F_j(\omega) \subseteq K_iA\}. \tag{9.3}$$

5 The collection of all subsets of Y is denoted by 2^Y.

Example 9.3 *(Continued)* Denote $A = \{\omega_p\}$, $B = \{\omega_{r,2}\}$, and $C = \{\omega_{r,1}, \omega_{r,2}\}$. Then

$$K_J A = \{\omega_p\} = A, \qquad K_J B = \emptyset, \qquad K_J C = \emptyset,$$
$$K_P A = \{\omega_p\} = A, \qquad K_P B = \{\omega_{r,2}\}, \qquad K_P C = \{\omega_{r,2}\}.$$

The content of the expression $K_P B = \{\omega_{r,2}\}$ is that only in state of the world $\omega_{r,2}$ does Paul know that event B obtains (meaning that only in that state of the world does he know that the car is red). The content of $K_J B = \emptyset$ is that there is no state of the world in which John knows that B obtains; i.e., he never knows that the car is red and that Paul is not color-blind. From this we conclude that

$$K_J K_P C = K_J B = \emptyset. \tag{9.4}$$

This means that there is no state of the world in which John knows that Paul knows that the car is red. In contrast, $\omega_p \in K_P K_J A$, which means that in state of the world ω_p Paul knows that John knows that the state of the world is ω_p (and in particular, that the car is purple). ◀

We can now present some simple results that follow from the above definition of knowledge. The first result states that if a player knows event A in state of the world ω, then it is necessarily true that $\omega \in A$. In other words, if a player knows the event A, then A necessarily obtains (because the true state of the world is contained within it).[6]

Theorem 9.9 $K_i A \subseteq A$ *for every event* $A \subseteq Y$ *and every player* $i \in N$.

Proof: Let $\omega \in K_i A$. From the definition of knowledge it follows that $F_i(\omega) \subseteq A$. Since $\omega \in F_i(\omega)$ it follows that $\omega \in A$, which is what we needed to prove. □

Our second result states that if event A is contained in event B, then the states of the world in which player i knows event A form a subset of the states of the world in which the player knows event B. In other words, in every state of the world in which a player knows event A, he also knows event B.

Theorem 9.10 *For every pair of events* $A, B \subseteq Y$, *and every player* $i \in N$,

$$A \subseteq B \quad \Longrightarrow \quad K_i A \subseteq K_i B. \tag{9.5}$$

Proof: We will show that $\omega \in K_i A$ implies that $\omega \in K_i B$. Suppose that $\omega \in K_i A$. By definition, $F_i(\omega) \subseteq A$, and because $A \subseteq B$, one has $F_i(\omega) \subseteq B$. Therefore, $\omega \in K_i B$, which is what we need to show. □

Our third result[7] says that if a player knows event A, then he knows that he knows event A, and conversely, if he knows that he knows event A, then he knows event A.

Theorem 9.11 *For every event* $A \subseteq Y$ *and every player* $i \in N$, *we have* $K_i K_i A = K_i A$.

Proof: Theorems 9.9 and 9.10 imply that $K_i K_i A \subseteq K_i A$. We will show that the opposite inclusion holds, namely, if $\omega \in K_i A$ then $\omega \in K_i K_i A$. If $\omega \in K_i A$ then $F_i(\omega) \subseteq A$. Therefore, for every $\omega' \in F_i(\omega)$, we have $\omega' \in F_i(\omega') = F_i(\omega) \subseteq A$. It follows that $\omega' \in K_i A$. As this is true for every $\omega' \in F_i(\omega)$, we deduce that $F_i(\omega) \subseteq K_i A$, which implies that $\omega \in K_i K_i A$. Thus, $K_i A \subseteq K_i K_i A$, which is what we wanted to prove. □

6 In the literature, this is known as the "axiom of knowledge."

7 One part of this theorem, namely, the fact that if a player knows an event, then he knows that he knows the event, is known in the literature as the "axiom of positive introspection."

More generally, the knowledge operator K_i of player i satisfies the following five properties, which collectively are called *Kripke's S5 System*:

1. $K_iY = Y$: the player knows that Y is the set of all states of the world.
2. $K_iA \cap K_iB = K_i(A \cap B)$: if the player knows event A and knows event B then he knows event $A \cap B$.
3. $K_iA \subseteq A$: if the player knows event A then event A obtains.
4. $K_iK_iA = K_iA$: if the player knows event A then he knows that he knows event A, and vice versa.
5. $(K_iA)^c = K_i((K_iA)^c)$: if the player does not know event A, then he knows that he does not know event A, and vice versa.[8,9]

Property 3 was proved in Theorem 9.9. Property 4 was proved in Theorem 9.11. The proof that the knowledge operator satisfies the other three properties is left to the reader (Exercise 9.1). In fact, Properties 1–5 characterize knowledge operators: for every operator $K : 2^Y \to 2^Y$ satisfying these properties there exists a partition $\mathcal{F}$ of Y that induces K via Equation (9.2) (Exercise 9.2).

Example 9.12 Anthony, Betty, and Carol are each wearing a hat. Hats may be red (r) or blue (b). Each one of the three sees the hats worn by the other two, but cannot see his or her own hat, and therefore does not know its color. This situation can be described by an Aumann model of incomplete information as follows:

- The set of players is $N = \{\text{Anthony, Betty, Carol}\}$.
- The set of states of nature is $S = \{(r,r,r),(r,r,b),(r,b,r),(r,b,b),(b,r,r),(b,r,b),(b,b,r),(b,b,b)\}$. A state of nature is described by three hat colors: that of Anthony's hat (the left letter), of Betty's hat (the middle letter), and of Carol (the right letter).
- The set of states of the world is $Y = \{\omega_{rrr}, \omega_{rrb}, \omega_{rbr}, \omega_{rbb}, \omega_{brr}, \omega_{brb}, \omega_{bbr}, \omega_{bbb}\}$.
- The function $\mathfrak{s} : Y \to S$ that maps every state of the world to a state of nature is defined by

$$\mathfrak{s}(\omega_{rrr}) = (r,r,r), \quad \mathfrak{s}(\omega_{rrb}) = (r,r,b), \quad \mathfrak{s}(\omega_{rbr}) = (r,b,r), \quad \mathfrak{s}(\omega_{rbb}) = (r,b,b),$$
$$\mathfrak{s}(\omega_{brr}) = (b,r,r), \quad \mathfrak{s}(\omega_{brb}) = (b,r,b), \quad \mathfrak{s}(\omega_{bbr}) = (b,b,r), \quad \mathfrak{s}(\omega_{bbb}) = (b,b,b).$$

The information partitions of Anthony, Betty, and Carol are as follows:

$$\mathcal{F}_A = \{\{\omega_{rrr}, \omega_{brr}\}, \{\omega_{rrb}, \omega_{brb}\}, \{\omega_{rbr}, \omega_{bbr}\}, \{\omega_{rbb}, \omega_{bbb}\}\}, \tag{9.6}$$
$$\mathcal{F}_B = \{\{\omega_{rrr}, \omega_{rbr}\}, \{\omega_{rrb}, \omega_{rbb}\}, \{\omega_{brr}, \omega_{bbr}\}, \{\omega_{brb}, \omega_{bbb}\}\}, \tag{9.7}$$
$$\mathcal{F}_C = \{\{\omega_{rrr}, \omega_{rrb}\}, \{\omega_{rbr}, \omega_{rbb}\}, \{\omega_{brr}, \omega_{brb}\}, \{\omega_{bbr}, \omega_{bbb}\}\}. \tag{9.8}$$

For example, when the state of the world is ω_{brb}, Anthony sees that Betty is wearing a red hat and that Carol is wearing a blue hat, but does not know whether his hat is red or blue, so that he knows that the state of the world is in the set $\{\omega_{rrb}, \omega_{brb}\}$, which is one of the elements of his

8 The first part of this property, i.e., the fact that if a player does not know an event, then he knows that he does not know it, is known in the literature as the "axiom of negative introspection."

9 For any event A, the complement of A is denoted by $A^c := Y \setminus A$.

partition $\mathcal{F}_A$. Similarly, if the state of the world is ω_{brb}, Betty knows that the state of the world is in her partition element $\{\omega_{brb}, \omega_{bbb}\}$, and Carol knows that the state of the world is in her partition element $\{\omega_{brr}, \omega_{brb}\}$.

Let R be the event "there is at least one red hat," that is,

$$R = \{\omega_{rrr}, \omega_{rrb}, \omega_{rbr}, \omega_{rbb}, \omega_{brr}, \omega_{brb}, \omega_{bbr}\}. \tag{9.9}$$

In which states of the world does Anthony know R? In which states does Betty know that Anthony knows R? In which states does Carol know that Betty knows that Anthony knows R? To begin answering the first question, note that in state of the world ω_{rrr}, Anthony knows R, because

$$F_A(\omega_{rrr}) = \{\omega_{rrr}, \omega_{brr}\} \subseteq R. \tag{9.10}$$

Anthony also knows R in each of the states of the world ω_{rrb}, ω_{rbr}, ω_{brb}, ω_{brr}, and ω_{bbr}. In contrast, in the states ω_{rbb} and ω_{bbb} he does not know R, because

$$F_A(\omega_{rbb}) = F_A(\omega_{bbb}) = \{\omega_{rbb}, \omega_{bbb}\} \not\subseteq R. \tag{9.11}$$

In summary,

$$K_A R = \{\omega \in Y : F_A(\omega) \subseteq R\} = \{\omega_{rrr}, \omega_{rbr}, \omega_{rrb}, \omega_{brb}, \omega_{brr}, \omega_{bbr}\}.$$

The analysis here is quite intuitive: Anthony knows R if either Betty or Carol (or both) is wearing a red hat, which occurs in the states of the world in the set $\{\omega_{rrr}, \omega_{rbr}, \omega_{rrb}, \omega_{brb}, \omega_{brr}, \omega_{bbr}\}$. When does Betty know that Anthony knows R? This requires calculating $K_B K_A R$.

$$\begin{aligned} K_B K_A R &= \{\omega \in Y : F_B(\omega) \subseteq K_A R\} \\ &= \{\omega \in Y : F_B(\omega) \subseteq \{\omega_{rrr}, \omega_{rbr}, \omega_{rrb}, \omega_{brb}, \omega_{brr}, \omega_{bbr}\}\} \\ &= \{\omega_{rrr}, \omega_{brr}, \omega_{rbr}, \omega_{bbr}\}. \end{aligned} \tag{9.12}$$

For example, since $F_B(\omega_{rbr}) = \{\omega_{rbr}, \omega_{rrr}\} \subseteq K_A R$ we conclude that $\omega_{rbr} \in K_B K_A R$. On the other hand, since $F_B(\omega_{brb}) = \{\omega_{brb}, \omega_{bbb}\} \not\subseteq K_A R$, it follows that $\omega_{brb} \not\in K_B K_A R$. The analysis here is once again intuitively clear: Betty knows that Anthony knows R only if Carol is wearing a red hat, which only occurs in the states of the world $\{\omega_{rrr}, \omega_{brr}, \omega_{rbr}, \omega_{bbr}\}$.

Finally, we answer the third question: when does Carol know that Betty knows that Anthony knows R? This requires calculating $K_C K_B K_A R$.

$$\begin{aligned} K_C K_B K_A R &= \{\omega \in Y : F_C(\omega) \subseteq K_B K_A R\} \\ &= \{\omega \in Y : F_C(\omega) \subseteq \{\omega_{rrr}, \omega_{brr}, \omega_{rbr}, \omega_{bbr}\}\} = \emptyset. \end{aligned} \tag{9.13}$$

For example, since $F_C(\omega_{rbr}) = \{\omega_{rbr}, \omega_{rbb}\} \not\subseteq K_B K_A R$, we conclude that $\omega_{rbr} \not\in K_C K_B K_A R$. In other words, there is no state of the world in which Carol knows that Betty knows that Anthony knows R. This is true intuitively, because as we saw previously, Betty knows that Anthony knows R only if Carol is wearing a red hat, but Carol does not know the color of her own hat.

This analysis enables us to conclude, for example, that in state of the world ω_{rrr} Anthony knows R, Betty knows that Anthony knows R, but Carol does not know that Betty knows that Anthony knows R. ◀

Note the distinction in Example 9.12 between states of nature and states of the world. The state of nature is the parameter with respect to which there is incomplete information: the colors of the hats worn by the three players. The state of the world includes in addition the mutual knowledge structure of the players regarding the state of nature. For example, the state of the world ω_{rrr} says a lot more than the fact that all three players are wearing red hats; for example, in this state of the world Carol knows there is at least one red hat, Carol knows that Anthony knows that there is at least one red hat, and Carol does not know that Betty knows that Anthony knows that there is at least one red hat. In Example 9.12 there is a one-to-one correspondence between the set of states of nature S and the set of states of the world Y. This is so since the mutual knowledge structure is uniquely determined by the configuration of the colors of the hats.

Example 9.13 Arthur, Harry, and Tom are in a room with two windows, one facing north and the other facing south. Two hats, one yellow and one brown, are placed on a table in the center of the room. After Harry and Tom leave the room, Arthur selects one of the hats and places it on his head. Tom and Harry peek in, each through a different window, watching Arthur (so that they both know the color of the hat Arthur is wearing). Neither Tom nor Harry knows whether or not the other player who has left the room is peeking through a window, and Arthur has no idea whether or not Tom or Harry is spying on him as he places one of the hats on his head. An Aumann model of incomplete information describing this situation is as follows:

- $N = \{\text{Arthur, Harry, Tom}\}$.
- $S = \{\text{Arthur wears the brown hat, Arthur wears the yellow hat}\}$.
- There are eight states of the world, each of which is designated by two indices: $Y = \{\omega_{b,\emptyset}, \omega_{b,T}, \omega_{b,H}, \omega_{b,TH}, \omega_{y,\emptyset}, \omega_{y,T}, \omega_{y,H}, \omega_{y,TH}\}$. The left index of ω indicates the color of the hat that Arthur is wearing (which is either brown or yellow), and the right index indicates which of the other players has been peeking into the room (Tom (T), Harry (H), both (TH), or neither($\emptyset$)).
- Arthur's partition contains two elements, because he knows the color of the hat on his head, but does not know who is peeking into the room: $\mathcal{F}_A = \{\{\omega_{b,\emptyset}, \omega_{b,H}, \omega_{b,T}, \omega_{b,TH}\}, \{\omega_{y,\emptyset}, \omega_{y,H}, \omega_{y,I}, \omega_{y,TH}\}\}$.
- Tom's partition contains three elements, one for each of his possible situations of information: Tom has not peeked into the room; Tom has peeked into the room and seen Arthur wearing the brown hat; Tom has peeked into the room and seen Arthur wearing the yellow hat. His partition is thus $\mathcal{F}_T = \{\{\omega_{b,\emptyset}, \omega_{b,H}, \omega_{y,\emptyset}, \omega_{y,H}\}, \{\omega_{b,T}, \omega_{b,TH}\}, \{\omega_{y,T}, \omega_{y,TH}\}\}$.

  For example, if Tom has peeked and seen the brown hat on Arthur's head, he knows that Arthur has selected the brown hat, but he does not know whether he is the only player who peeked (corresponding to the state of the world $\omega_{b,T}$) or whether Harry has also peeked (state of the world $\omega_{b,TH}$).
- Similarly, Harry's partition is $\mathcal{F}_H = \{\{\omega_{b,\emptyset}, \omega_{b,T}, \omega_{y,\emptyset}, \omega_{y,T}\}, \{\omega_{b,H}, \omega_{b,TH}\}, \{\omega_{y,H}, \omega_{y,TH}\}\}$.
- The function s is defined by

$$s(\omega_{b,\emptyset}) = s(\omega_{b,T}) = s(\omega_{b,H}) = s(\omega_{b,TH}) = \text{Arthur wears the brown hat;}$$
$$s(\omega_{y,\emptyset}) = s(\omega_{y,T}) = s(\omega_{y,H}) = s(\omega_{y,TH}) = \text{Arthur wears the yellow hat.}$$

In this model, for example, if the true state of the world is $\omega_* = \omega_{b,TH}$, then Arthur is wearing the brown hat, and both Tom and Harry have peeked into the room. The event "Arthur is wearing the brown hat" is $B = \{\omega_{b,\emptyset}, \omega_{b,T}, \omega_{b,H}, \omega_{b,TH}\}$. Tom and Harry know that Arthur's hat is brown only if they have peeked into the room. Therefore,

$$K_T B = \{\omega_{b,T}, \omega_{b,TH}\}, \qquad K_H B = \{\omega_{b,H}, \omega_{b,TH}\}. \tag{9.14}$$

Given Equation (9.14), since the set $K_H B$ is not included in any of the elements in Tom's partition, we conclude that $K_T K_H B = \emptyset$. In other words, in any state of the world, Tom does not know whether or not Harry knows that Arthur is wearing the brown hat, and therefore, in particular, this is the case at the given state of the world, $\omega_{b,TH}$. We similarly conclude that $K_H K_T B = \emptyset$: in any state of the world, Harry does not know that Tom knows that Arthur is wearing the brown hat (and in particular this is the case at the true state of the world, $\omega_{b,TH}$). This is all quite intuitive; Tom knows that Arthur is wearing the brown hat only if he has peeked into the room, but Harry does not know whether or not Tom has peeked into the room.

Note again the distinction between a state of nature and a state of the world. The objective fact about which the players have incomplete information is the color of the hat atop Arthur's head. Each one of the four states of the world $\{\omega_{y,\emptyset}, \omega_{y,H}, \omega_{y,T}, \omega_{y,TH}\}$ corresponds to the state of nature "Arthur wears the yellow hat," yet they differ in the knowledge that the players have regarding the state of nature. In the state of the world $\omega_{y,\emptyset}$, Arthur wears the yellow hat, but Tom and Harry do not know that, while in state of the world $\omega_{y,H}$, Arthur wears the yellow hat and Harry knows that, but Tom does not know that. Note that in both of these states of the world Tom and Arthur do not know that Harry knows the color of Arthur's hat, Harry and Arthur do not know whether or not Tom knows the color of the hat, and in each state of the world there are additional statements that can be made regarding the players' mutual knowledge of Arthur's hat. ◀

The insights gleaned from these examples can be formulated and proven rigorously.

Definition 9.14 *A* knowledge hierarchy *among players in state of the world ω over the set of states of the world Y is a system of "yes" or "no" answers to each question of the form "in a state of the world ω, does player i_1 know that player i_2 knows that player i_3 knows... that player i_l knows event A"? for any event $A \subseteq Y$ and any finite sequence $i_1, i_2, \ldots, i_l$ of players*[10] *in N.*

The answer to the question "does player i_1 know that player i_2 knows that player i_3 knows... that player i_l knows event A?" in a state of the world ω is affirmative if $\omega \in K_{i_1} K_{i_2} \cdots K_{i_l} A$, and negative if $\omega \notin K_{i_1} K_{i_2} \cdots K_{i_l} A$. Since for every event A and every sequence of players $i_1, i_2, \ldots, i_l$ the event $K_{i_1} K_{i_2} \cdots K_{i_l} A$ is well defined and calculable in an Aumann model of incomplete information, every state of the world defines a knowledge hierarchy. We have therefore derived the following theorem.

Theorem 9.15 *Every situation of incomplete information $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \omega_*)$ uniquely determines a knowledge hierarchy over the set of states of the world Y in state of the world ω_*.*

For every subset $C \subseteq S$ of the set of states of nature, we can consider the event that contains all states of the world whose state of nature is an element of C:

$$\mathfrak{s}^{-1}(C) := \{\omega \in Y : \mathfrak{s}(\omega) \in C\}. \tag{9.15}$$

For example, in Example 9.13 the set of states of nature {yellow} corresponds to the event $\{\omega_{y,\emptyset}, \omega_{y,H}, \omega_{y,G}, \omega_{y,TH}\}$ in Y. Every subset of S is called an *event in S*. We define knowledge of events in S as follows: in a state of the world ω player i knows event C in S if and only if he knows the event $\mathfrak{s}^{-1}(C)$, i.e., if and only if $\omega \in K_i(\mathfrak{s}^{-1}(C))$. In the same manner, in state of the world ω player i_1 knows that player i_2 knows that player i_3 knows... that player i_l knows event C in S if and only if in state of the world ω player i_1 knows that player i_2 knows that player i_3 knows ... that player i_l knows $\mathfrak{s}^{-1}(C)$.

10 A player may appear several times in the chain $i_1, i_2, \ldots, i_l$. For example, the chain player 2 knows that player 1 knows that player 3 knows that player 2 knows event A is a legitimate chain.

Corollary 9.16 is a consequence of Theorem 9.15 (Exercise 9.10).

Corollary 9.16 *Every situation of incomplete information* $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \omega_*)$ *uniquely determines a knowledge hierarchy over the set of states of nature S in state of the world* ω_*.

Having defined the knowledge operators of the players, we next turn to the definition of the concept of common knowledge, which was previously defined informally (see Definition 9.2).

Definition 9.17 *Let* $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ *be an Aumann model of incomplete information, let* $A \subseteq Y$ *be an event, and let* $\omega \in Y$ *be a state of the world. The event A is* common knowledge *in* ω *if for every finite sequence of players* $i_1, i_2, \ldots, i_l$,

$$\omega \in K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A. \tag{9.16}$$

That is, event A is common knowledge at state of the world ω if in ω every player knows event A, every player knows that every player knows event A, etc. In Examples 9.12 and 9.13 the only event that is common knowledge in any state of the world is Y (Exercise 9.12). In Example 9.3 (page 322) the event $\{\omega_{\mathrm{p}}\}$ (and every event containing it) is common knowledge in state of the world ω_{p}, and the event $\{\omega_{\mathrm{g},1}, \omega_{\mathrm{g},2}, \omega_{\mathrm{r},1}, \omega_{\mathrm{r},2}\}$ (and the event Y containing it) is common knowledge in every state of the world contained in this event.

Example 9.18 Abraham selects an integer from the set $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. He tells Jefferson whether the number he has selected is even or odd, and tells Ulysses the remainder left over from dividing that number by 4. The corresponding Aumann model of incomplete information depicting the induced situation of Jefferson and Ulysses is:

- $N = \{\text{Jefferson, Ulysses}\}$.
- $S = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$: the state of nature is the number selected by Abraham.
- $Y = \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}, \omega_{14}\}$.
- The function $\mathfrak{s} : Y \to S$ is given by $\mathfrak{s}(\omega_k) = k$ for every $k \in S$.
- Since Jefferson knows whether the number is even or odd, his partition contains two elements, corresponding to the subset of even numbers and the subset of odd numbers in the set Y:

$$\mathcal{F}_{\mathrm{J}} = \{\{\omega_5, \omega_7, \omega_9, \omega_{11}, \omega_{13}\}, \{\omega_6, \omega_8, \omega_{10}, \omega_{12}, \omega_{14}\}\}. \tag{9.17}$$

- As Ulysses knows the remainder left over from dividing the number by 4, his partition contains four elements, one for each possible remainder:

$$\mathcal{F}_{\mathrm{U}} = \{\{\omega_8, \omega_{12}\}, \{\omega_5, \omega_9, \omega_{13}\}, \{\omega_6, \omega_{10}, \omega_{14}\}, \{\omega_7, \omega_{11}\}\}. \tag{9.18}$$

In the state of the world ω_6, the event that the selected number is even, i.e., $A = \{\omega_6, \omega_8, \omega_{10}, \omega_{12}, \omega_{14}\}$, is common knowledge. Indeed, $K_{\mathrm{J}}A = K_{\mathrm{U}}A = A$, and therefore it follows that $K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A = A$ for every finite sequence of players $i_1, i_2, \ldots, i_l$. Since $\omega_6 \in A$, it follows from Definition 9.17 that in state of the world ω_6 the event A is common knowledge among Jefferson and Ulysses. Similarly, in state of the world ω_9, the event that the selected number is odd, $B = \{\omega_5, \omega_7, \omega_9, \omega_{11}, \omega_{13}\}$, is common knowledge among Jefferson and Ulysses (verify!). ◄

Remark 9.19 *From Definition 9.17 and Theorem 9.10 we conclude that if event A is common knowledge in a state of the world ω, then every event containing A is also common knowledge in ω.* ♦

Remark 9.20 *The definition of common knowledge can be expanded to events in S: an event C in S is common knowledge in a state of the world ω if the event $\mathfrak{s}^{-1}(C)$ is common knowledge in ω. For example, in Example 9.13 in state of the world $\omega_{b,TH}$ the event (in the set of states of nature) "Arthur selects the brown hat" is not common knowledge among the players (verify!).* ♦

Remark 9.21 *If event A is common knowledge in a state of the world ω, then in particular $\omega \in K_i A$ and so $F_i(\omega) \subseteq A$ for each $i \in N$. In other words, all players know A in ω.* ♦

Remark 9.22 *We can also speak of common knowledge among a subset of the players $M \subseteq N$: in a state of the world ω, event A is common knowledge among the players in M if Equation (9.16) is satisfied for any finite sequence $i_1, i_2, \ldots, i_l$ of players in M.* ♦

Theorem 9.23 states that if there is a player who cannot distinguish between ω and ω', then every event that is common knowledge in ω is also common knowledge in ω'.

Theorem 9.23 *If event A is common knowledge in state of the world ω, and if $\omega' \in F_i(\omega)$ for some player $i \in N$, then the event A is also common knowledge in state of the world ω'.*

Proof: Suppose that $\omega' \in F_i(\omega)$ for some player $i \in N$. As the event A is common knowledge in ω, for any sequence $i_1, i_2, \ldots, i_l$ of players we have

$$\omega \in K_i K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A. \tag{9.19}$$

Remark 9.21 implies that

$$F_i(\omega) \subseteq K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A. \tag{9.20}$$

Since $\omega' \in F_i(\omega') = F_i(\omega)$ it follows that $\omega' \in K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A$. As this is true for any sequence $i_1, i_2, \ldots, i_l$ of players, the event A is common knowledge in ω'. □

We next turn to characterizing sets that are common knowledge. Given an Aumann model of incomplete information $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$, define the graph $G = (Y, V)$ in which the set of vertices is the set of states of the world Y, and there is an edge between vertices ω and ω' if and only if there is a player i such that $\omega' \in F_i(\omega)$. Note that the condition defining the edges of the graph is symmetric: $\omega' \in F_i(\omega)$ if and only if $F_i(\omega) = F_i(\omega')$, if and only if $\omega \in F_i(\omega')$; hence $G = (Y, V)$ is an undirected graph.

A set of vertices C in a graph is a *connected component* if the following two conditions are satisfied:

- For every $\omega, \omega' \in C$, there exists a path connecting ω with ω', i.e., there exist $\omega = \omega_1, \omega_2, \ldots, \omega_K = \omega'$ such that for each $k = 1, 2, \ldots, K-1$ the graph contains an edge connecting ω_k and ω_{k+1}.
- There is no edge connecting a vertex in C with a vertex that is not in C.

The *connected component of* ω in the graph, denoted by $C(\omega)$, is the (unique) connected component containing ω.

Theorem 9.24 *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ be an Aumann model of incomplete information and let G be the graph corresponding to this model. Let $\omega \in Y$ be a state of the world and let $A \subseteq Y$ be an event. Then event A is common knowledge in state of the world ω if and only if $A \supseteq C(\omega)$.*

Proof: First we prove that if A is common knowledge in ω, then $C(\omega) \subseteq A$. Suppose then that $\omega' \in C(\omega)$. We want to show that $\omega' \in A$. From the definition of a connected component, there is a path connecting ω with ω'; we denote that path by $\omega = \omega_1, \omega_2, \ldots, \omega_K = \omega'$. We prove by induction on k that $\omega_k \in A$, and that A is common knowledge in ω_k, for every $1 \le k \le K$. For $k = 1$, because the event A is common knowledge in ω, we deduce that $\omega_1 = \omega \in A$. Suppose now that $\omega_k \in A$ and A is common knowledge in ω_k. We will show that $\omega_{k+1} \in A$ and that A is common knowledge in ω_{k+1}. Because there is an edge connecting ω_k and ω_{k+1}, there is a player i such that $\omega_{k+1} \in F_i(\omega_k)$. It follows from Theorem 9.23 that the event A is common knowledge in ω_{k+1}. From Remark 9.21 we conclude that $\omega_{k+1} \in A$. This completes the inductive step, so that in particular $\omega' = \omega_K \in A$.

Consider now the other direction: if $C(\omega) \subseteq A$, then event A is common knowledge in state of the world ω. To prove this, it suffices to show that $C(\omega)$ is common knowledge in ω, because from Remark 9.19 it will then follow that any event containing $C(\omega)$, and in particular A, is also common knowledge in ω. Let i be a player in N. Because $C(\omega)$ is a connected component of G, for each $\omega' \in C(\omega)$, we have $F_i(\omega') \subseteq C(\omega)$. It follows that

$$C(\omega) \supseteq \bigcup_{\omega' \in C(\omega)} F_i(\omega') \supseteq \bigcup_{\omega' \in C(\omega)} \{\omega'\} = C(\omega). \tag{9.21}$$

In other words, for each player i the set $C(\omega)$ is the union of all the elements of $\mathcal{F}_i$ contained in it. This implies that $K_i(C(\omega)) = C(\omega)$. As this is true for every player $i \in N$, it follows that for every sequence of players $i_1, i_2, \ldots, i_l$,

$$\omega \in C(\omega) = K_{i_1} K_{i_2} \cdots K_{i_l} C(\omega), \tag{9.22}$$

and therefore $C(\omega)$ is common knowledge in ω. □

The following corollary follows from Theorem 9.24 and Remark 9.19.

Corollary 9.25 *In every state of the world $\omega \in Y$, the event $C(\omega)$ is common knowledge among the players, and it is the smallest event that is common knowledge in ω.*

For this reason, $C(\omega)$ is sometimes called the *common knowledge component* among the players in state of the world ω.

Remark 9.26 *The proof of Theorem 9.24 shows that for each player $i \in N$, the set $C(\omega)$ is the union of the elements of $\mathcal{F}_i$ contained in it, and it is the smallest event containing ω that satisfies this property. The set of all the connected components of the graph G defines a partition of Y, which is called the* meet *of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$. This is the finest partition that satisfies the property that each partition $\mathcal{F}_i$ is a refinement of it. We can therefore formulate Theorem 9.24 equivalently as follows. Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ be* *an Aumann model of incomplete information. Event A is common knowledge in state of the world $\omega \in Y$ if and only if A contains the element of the meet containing ω.* ♦

9.2 The Aumann model of incomplete information with beliefs

The following model extends the Aumann model of incomplete information presented in the previous section.

Definition 9.27 *An* Aumann model of incomplete information with beliefs *(over a set of states of nature S) consists of five elements $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$, where:*

- *N is a finite set of players;*
- *Y is a finite set of states of the world;*
- *$\mathcal{F}_i$ is a partition of Y, for each $i \in N$;*
- *$\mathfrak{s} : Y \to S$ is a function associating a state of nature to every state of the world;*
- *$\mathbf{P}$ is a probability distribution over Y such that $\mathbf{P}(\omega) > 0$ for each $\omega \in Y$.*

Comparing this definition to that of the Aumann model of incomplete information (Definition 9.4), we have added one new element, namely, the probability distribution $\mathbf{P}$ over Y, which is called the *common prior*. In this model, a state of the world ω_* is selected by a random process in accordance with the common prior probability distribution $\mathbf{P}$. After the true state of the world has been selected by this random process, each player i learns his partition element $F_i(\omega_*)$ that contains ω_*. Prior to the stage at which private information is revealed, the players share a common prior distribution, which is interpreted as their belief about the probability that any specific state of the world in Y is the true one. After each player i has acquired his private information $F_i(\omega_*)$, he updates his beliefs. This process of belief updating is the main topic of this section.

The assumption that all the players share a common prior is a strong assumption, and in many cases there are good reasons to doubt that it obtains. We will return to this point later in the chapter. In contrast, the assumption that $\mathbf{P}(\omega) > 0$ for all $\omega \in Y$ is not a strong assumption. As we will show, a state of the world ω for which $\mathbf{P}(\omega) = 0$ is one to which all the players assign probability 0, and it can be removed from consideration in Y.

In the following examples and in the rest of this chapter, whenever the states of nature are irrelevant we will specify neither the set S nor the function $\mathfrak{s}$.

Example 9.28 Consider the following Aumann model:

- The set of players is $N = \{\mathrm{I}, \mathrm{II}\}$.
- The set of states of the world is $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The information partitions of the players are

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}. \tag{9.23}$$

- The common prior $\mathbf{P}$ is

$$\mathbf{P}(\omega_1) = \tfrac{1}{4}, \quad \mathbf{P}(\omega_2) = \tfrac{1}{4}, \quad \mathbf{P}(\omega_3) = \tfrac{1}{3}, \quad \mathbf{P}(\omega_4) = \tfrac{1}{6}. \tag{9.24}$$

A graphic representation of the players' partitions and the prior probability distribution is provided in Figure 9.2. Player I's partition elements are marked by a solid line, while Player II's partition elements are denoted by a dotted line.

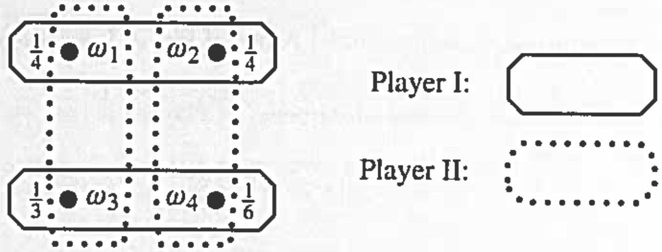


Figure 9.2 The information partitions and the prior distribution in Example 9.28

What are the beliefs of each player about the state of the world? Prior to the chance move that selects the state of the world, the players have a common prior distribution over the states of the world. When a player receives information that indicates that the true state of the world is in the partition element $F_i(\omega_*)$, he updates his beliefs about the states of the world by calculating the conditional probability given his information. For example, if the state of the world is ω_1, Player I knows that the state of the world is either ω_1 or ω_2. Player I's beliefs are therefore

$$\mathbf{P}(\omega_1 \mid \{\omega_1, \omega_2\}) = \frac{p(\omega_1)}{p(\omega_1) + p(\omega_2)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}, \tag{9.25}$$

and similarly

$$\mathbf{P}(\omega_2 \mid \{\omega_1, \omega_2\}) = \frac{p(\omega_2)}{p(\omega_1) + p(\omega_2)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}. \tag{9.26}$$

In words, if Player I's information is that the state of the world is in $\{\omega_1, \omega_2\}$, he attributes probability $\frac{1}{2}$ to the state of the world ω_1 and probability $\frac{1}{2}$ to the state of the world ω_2. The tables appearing in Figure 9.3 are arrived at through a similar calculation. The upper table describes Player I's beliefs, as a function of his information partition, and the lower table represents Player II's beliefs as a function of his information partition.

Player I's beliefs:

Player I's Information	ω_1	ω_2	ω_3	ω_4
$\{\omega_1, \omega_2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\{\omega_3, \omega_4\}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$

Player II's beliefs:

Player II's Information	ω_1	ω_2	ω_3	ω_4
$\{\omega_1, \omega_3\}$	$\frac{3}{7}$	0	$\frac{4}{7}$	0
$\{\omega_2, \omega_4\}$	0	$\frac{3}{5}$	0	$\frac{2}{5}$

Figure 9.3 The beliefs of the players in Example 9.28

For example, if Player II's information is $\{\omega_2, \omega_4\}$ (i.e., the state of the world is either ω_2 or ω_4), he attributes probability $\frac{3}{5}$ to the state of the world ω_2 and probability $\frac{2}{5}$ to the state of the world ω_4.

A player's beliefs will be denoted by square brackets in which states of the world appear alongside the probabilities that are ascribed to them. For example, $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$ represents beliefs in which probability $\frac{3}{5}$ is ascribed to state of the world ω_2, and probability $\frac{2}{5}$ is ascribed to state of the world

ω_4. The calculations performed above yield the first-order beliefs of the players at all possible states of the world. These beliefs can be summarized as follows:

- In state of the world ω_1 the first-order belief of Player I is $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$ and that of Player II is $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$.
- In state of the world ω_2 the first-order belief of Player I is $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$ and that of Player II is $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.
- In state of the world ω_3 the first-order belief of Player I is $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$ and that of Player II is $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$.
- In state of the world ω_4 the first-order belief of Player I is $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$ and that of Player II is $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.

Given the first-order beliefs of the players over Y, we can construct the second-order beliefs, by which we mean the beliefs each player has about the state of the world and the first-order beliefs of the other player. In state of the world ω_1 (or ω_2) Player I attributes probability $\frac{1}{2}$ to the state of the world being ω_1 and probability $\frac{1}{2}$ to the state of the world being ω_2. As we noted above, when the state of the world is ω_1, the first-order belief of Player II is $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$, and when the state of the world is ω_2, Player II's first-order belief is $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$. Therefore:

- In state of the world ω_1 (or ω_2) Player I attributes probability $\frac{1}{2}$ to the state of the world being ω_1 and the first-order belief of Player II being $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$, and probability $\frac{1}{2}$ to the state of the world being ω_2 and Player II's first-order belief being $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.

We can similarly calculate the second-order beliefs of each of the players in each state of the world:

- In state of the world ω_3 (or ω_4) Player I attributes probability $\frac{2}{3}$ to the state of the world being ω_3 and the first-order belief of Player II being $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$, and probability $\frac{1}{3}$ to the state of the world being ω_4 and Player II's first-order belief being $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.
- In state of the world ω_1 (or ω_3) Player II attributes probability $\frac{3}{7}$ to the state of the world being ω_1 and the first-order belief of Player I being $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$, and probability $\frac{4}{7}$ to the state of the world being ω_3 and Player I's first-order belief being $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$.
- In state of the world ω_2 (or ω_4) Player II attributes probability $\frac{3}{5}$ to the state of the world being ω_2 and the first-order belief of Player I being $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$, and probability $\frac{2}{5}$ to the state of the world being ω_4 and Player I's first-order belief being $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$.

These calculations can be continued to arbitrarily high orders in a similar manner to yield belief hierarchies of the two players. ◀

Theorem 9.29 says that in an Aumann model, knowledge is equivalent to belief with probability 1. The theorem, however, requires assuming that $\mathbf{P}(\omega) > 0$ for each $\omega \in Y$; without that assumption the theorem's conclusion does not obtain (Exercise 9.21). In Example 9.36 we will see that the conclusion of the theorem also fails to hold when the set of states of the world is infinite.

Theorem 9.29 *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs. Then for each $\omega \in Y$, for each player $i \in N$, and for every event $A \subseteq Y$, player i knows event A in state of the world ω if and only if he attributes probability 1 to that event:*

$$\mathbf{P}(A \mid F_i(\omega)) = 1 \iff F_i(\omega) \subseteq A. \tag{9.27}$$

Notice that the assumption that $\mathbf{P}(\omega) > 0$ for every $\omega \in Y$, together with $\omega \in F_i(\omega)$ for every $\omega \in Y$, yields $\mathbf{P}(F_i(\omega)) > 0$ for each player $i \in N$ and every state of the world $\omega \in Y$, so that the conditional probability in Equation (9.27) is well defined.

Proof: Suppose first that $F_i(\omega) \subseteq A$. Then

$$\mathbf{P}(A \mid F_i(\omega)) \geq \mathbf{P}(F_i(\omega) \mid F_i(\omega)) = 1, \tag{9.28}$$

so that $\mathbf{P}(A \mid F_i(\omega)) = 1$. To prove the reverse implication, if $\mathbf{P}(A \mid F_i(\omega)) = 1$ then

$$\mathbf{P}(A \mid F_i(\omega)) = \frac{\mathbf{P}(A \cap F_i(\omega))}{\mathbf{P}(F_i(\omega))} = 1, \tag{9.29}$$

which yields $\mathbf{P}(A \cap F_i(\omega)) = \mathbf{P}(F_i(\omega))$. From the assumption that $\mathbf{P}(\omega') > 0$ for each $\omega' \in Y$ we conclude that $A \cap F_i(\omega) = F_i(\omega)$, that is, $F_i(\omega) \subseteq A$. □

A *situation of incomplete information with beliefs* is a vector $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P}, \omega_*)$ composed of an Aumann model of incomplete information with beliefs $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ together with a state of the world $\omega_* \in Y$. The next theorem follows naturally from the analysis we performed in Example 9.28, and it generalizes Theorem 9.15 and Corollary 9.16 to situations of belief.

Theorem 9.30 *Every situation of incomplete information with beliefs $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P}, \omega_*)$ uniquely determines a mutual belief hierarchy among the players over the states of the world Y, and therefore also a mutual belief hierarchy over the states of nature S.*

The above formulation is not precise, as we have not formally defined what the term "mutual belief hierarchy" means. The formal definition is presented in Chapter 11 where we will show that each state of the world is in fact a pair, consisting of a state of nature and a mutual belief hierarchy among the players over the states of nature S. The inductive description of belief hierarchies, as presented in the examples above and the examples below, will suffice for this chapter.

In Example 9.28 we calculated the belief hierarchy of the players in each state of the world. A similar calculation can be performed with respect to events.

Example 9.28 *(Continued)* Consider the situation in which $\omega_* = \omega_1$ and the event $A = \{\omega_2, \omega_3\}$. As Player I's information in state of the world ω_1 is $\{\omega_1, \omega_2\}$, the conditional probability that he ascribes to event A in state of the world ω_1 (or ω_2) is

$$\mathbf{P}(A \mid \{\omega_1, \omega_2\}) = \frac{\mathbf{P}(A \cap \{\omega_1, \omega_2\})}{\mathbf{P}(\{\omega_1, \omega_2\})} = \frac{\mathbf{P}(\{\omega_1\})}{\mathbf{P}(\{\omega_1, \omega_2\})} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}. \tag{9.30}$$

Because Player II's information in state of the world ω_1 is $\{\omega_1, \omega_3\}$, the conditional probability that he ascribes to event A in state of the world ω_1 (or ω_3) is

$$\mathbf{P}(A \mid \{\omega_1, \omega_3\}) = \frac{\mathbf{P}(A \cap \{\omega_1, \omega_3\})}{\mathbf{P}(\{\omega_1, \omega_3\})} = \frac{\mathbf{P}(\{\omega_3\})}{\mathbf{P}(\{\omega_1, \omega_3\})} = \frac{\frac{1}{3}}{\frac{1}{4} + \frac{1}{3}} = \frac{4}{7}. \tag{9.31}$$

Second-order beliefs can also be calculated readily. In state of the world ω_1, Player I ascribes probability $\frac{1}{2}$ to the true state being ω_1, in which case the probability that Player II ascribes to event A is $\frac{4}{7}$; he ascribes probability $\frac{1}{2}$ to the true state being ω_2, in which case the probability that Player II ascribes to event A is $(\frac{1}{4})/(\frac{1}{4}+\frac{1}{6}) = \frac{3}{5}$. These are Player I's second-order beliefs about event A in state of the world ω_1. We can similarly calculate the second-order beliefs of Player II, as well as all the higher-order beliefs of the two players. ◀

Example 9.31 Consider again the Aumann model of incomplete information with beliefs presented in Example 9.28, but now with the common prior given by

$$\mathbf{P}(\omega_1) = \mathbf{P}(\omega_4) = \tfrac{1}{6}, \quad \mathbf{P}(\omega_2) = \mathbf{P}(\omega_3) = \tfrac{1}{3}. \tag{9.32}$$

The partitions $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are graphically depicted in Figure 9.4.

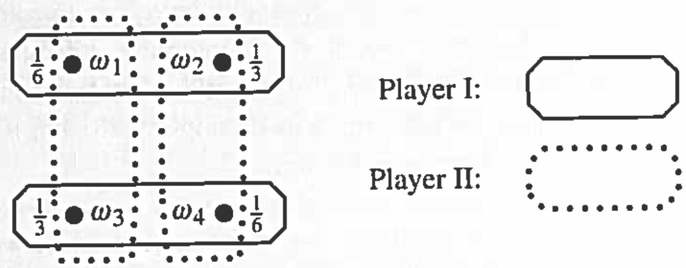


Figure 9.4 The information partitions and the prior distribution in Example 9.31

Since $\omega_1 \in F_{\mathrm{I}}(\omega_2)$, $\omega_2 \in F_{\mathrm{II}}(\omega_4)$, and $\omega_4 \in F_{\mathrm{I}}(\omega_3)$ in the graph corresponding to this Aumann model, all states in Y are connected. Hence the only connected component in the graph is Y (verify!), and therefore the only event that is common knowledge in any state of the world ω is Y (Theorem 9.24). Consider now the event $A = \{\omega_2, \omega_3\}$ and the situation in which $\omega_* = \omega_1$. What is the conditional probability that the players ascribe to A? Similarly to the calculation performed in Example 9.28,

$$\mathbf{P}(A \mid \{\omega_1, \omega_2\}) = \frac{\mathbf{P}(A \cap \{\omega_1, \omega_2\})}{\mathbf{P}(\{\omega_1, \omega_2\})} = \frac{\mathbf{P}(\{\omega_2\})}{\mathbf{P}(\{\omega_1, \omega_2\})} = \frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = \frac{2}{3}, \tag{9.33}$$

and we can also readily calculate that both players ascribe probability $\frac{2}{3}$ to event A in each state of the world. Formally:

$$\left\{\omega \colon q_{\mathrm{I}} := \mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{2}{3}\right\} = Y, \qquad \left\{\omega \colon q_{\mathrm{II}} := \mathbf{P}(A \mid F_{\mathrm{II}}(\omega)) = \tfrac{2}{3}\right\} = Y. \tag{9.34}$$

It follows from the definition of the knowledge operator that the event "Player I ascribes probability $\frac{2}{3}$ to A" is common knowledge in each state of the world, and the event "Player II ascribes probability $\frac{2}{3}$ to A" is also common knowledge in each state of the world. In other words, in this situation the probabilities that the two players ascribe to event A are both common knowledge and equal to each other. ◀

Is it a coincidence that the probabilities q_{I} and q_{II} that the two players assign to the event A in Example 9.31 are equal (both being $\frac{2}{3}$)? Can there be a situation in which it is common knowledge that to the event A, Player I ascribes probability q_{I} and Player II ascribes probability q_{II}, where $q_{\mathrm{I}} \neq q_{\mathrm{II}}$? Theorem 9.32 asserts that this state of affairs is impossible.

Theorem 9.32 Aumann's Agreement Theorem (Aumann [1976]) *Let $(N, Y, (\mathcal{F}_i)_{i \in N}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs, and suppose that $n = 2$ (i.e., there are two players). Let $A \subseteq Y$ be an event and let $\omega \in Y$ be a state of the world. If the event "Player I ascribes probability q_{I} to A" is common knowledge in ω, and the event "Player II ascribes probability q_{II} to A" is also common knowledge in ω, then $q_{\mathrm{I}} = q_{\mathrm{II}}$.*

Let us take a moment to consider the significance of this theorem before proceeding to its proof. The theorem states that if two players begin with "identical beliefs about the world" (represented by the common prior $\mathbf{P}$) but receive disparate information (represented by their respective partition elements containing ω), then "they cannot agree to disagree": if they agree that the probability that Player I ascribes to a particular event is q_{I}, then they cannot also agree that Player II ascribes a probability q_{II} to the same event, unless $q_{\mathrm{I}} = q_{\mathrm{II}}$. If they disagree regarding a particular fact (for example, Player I ascribes probability q_{I} to event A and Player II ascribes probability q_{II} to the same event), then the fact that they disagree cannot be common knowledge. Since we know that people often agree to disagree, we must conclude that either (a) different people begin with different prior distributions over the states of the world, or (b) people incorrectly calculate conditional probabilities when they receive information regarding the true state of the world.

Proof of Theorem 9.32: Let C be the connected component of ω in the graph corresponding to the given Aumann model. It follows from Theorem 9.24 that event C is common knowledge in state of the world ω. The event C can be represented as a union of partition elements in $\mathcal{F}_{\mathrm{I}}$; that is, $C = \bigcup_j F_{\mathrm{I}}^j$, where $F_{\mathrm{I}}^j \in \mathcal{F}_{\mathrm{I}}$ for each j. Since $\mathbf{P}(\omega') > 0$ for every $\omega' \in Y$, it follows that $\mathbf{P}(F_{\mathrm{I}}^j) > 0$ for every j, and therefore $\mathbf{P}(C) > 0$.

The fact that Player I ascribes probability q_{I} to the event A is common knowledge in ω. It follows that the event A contains the event C (Corollary 9.25), and therefore each one of the events $(F_{\mathrm{I}}^j)_j$. This implies that for each of the sets F_{I}^j the conditional probability of A, given that Player I's information is F_{I}^j, equals q_{I}. In other words, for each j,

$$\mathbf{P}(A \mid F_{\mathrm{I}}^j) = \frac{\mathbf{P}(A \cap F_{\mathrm{I}}^j)}{\mathbf{P}(F_{\mathrm{I}}^j)} = q_{\mathrm{I}}. \tag{9.35}$$

As this equality holds for every j, and $C = \bigcup_j F_{\mathrm{I}}^j$, it follows from Equation (9.35) that

$$\mathbf{P}(A \cap C) = \sum_j \mathbf{P}(A \cap F_{\mathrm{I}}^j) = q_{\mathrm{I}} \sum_j \mathbf{P}(F_{\mathrm{I}}^j) = q_{\mathrm{I}}\mathbf{P}(C). \tag{9.36}$$

We similarly derive that

$$\mathbf{P}(A \cap C) = q_{\mathrm{II}}\mathbf{P}(C). \tag{9.37}$$

Finally, since $\mathbf{P}(C) > 0$, Equations (9.36) and (9.37) imply that $q_{\mathrm{I}} = q_{\mathrm{II}}$, which is what we wanted to show. □

How do players arrive at a situation in which the probabilities q_{I} and q_{II} that they ascribe to a particular event A are common knowledge? In Example 9.31, each player calculates the conditional probability of A given a partition element of the other player, and comes to the conclusion that no matter which partition element of the other player is used for the conditioning, the conditional probability turns out to be the same. That is why q_i is common knowledge among the players for $i = \mathrm{I}, \mathrm{II}$.

In most cases the conditional probability of an event is not common knowledge, because it varies from one partition element to another. We can, however, describe a process of information transmission between the players that guarantees that these conditional probabilities will become common knowledge when the process is complete (see Exercises 9.25 and 9.26). Suppose that each player publicly announces the conditional probability he ascribes to event A given the information (i.e., the partition element) at his disposal. After each player has heard the other player's announcement, he can rule out some states of the world, because they are impossible: possible states of the world are only those in which the conditional probability that the other player ascribes to event A is the conditional probability that he publicly announced. Each player can then update the conditional probability that he ascribes to event A following the elimination of impossible states of the world, and again publicly announce the new conditional probability he has calculated. Following this announcement, the players can again rule out the states of the world in which the updated conditional probability of the other player differs from that which he announced, update their conditional probabilities, and announce them publicly. This can be repeated again and again. Using Aumann's Agreement Theorem (Theorem 9.32), it can be shown that at the end of this process the players will converge to the same conditional probability, which will be common knowledge among them (Exercise 9.28).

Example 9.33 We provide now an example of the dynamic process just described. More examples can be found in Exercises 9.25 and 9.26. Consider the following Aumann model of incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The information partitions of the players are

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}. \tag{9.38}$$

- The prior distribution is

$$\mathbf{P}_{\mathrm{II}}(\omega_1) = \mathbf{P}_{\mathrm{II}}(\omega_4) = \tfrac{1}{3}, \quad \mathbf{P}_{\mathrm{II}}(\omega_2) = \mathbf{P}_{\mathrm{II}}(\omega_3) = \tfrac{1}{6}. \tag{9.39}$$

The partition elements $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are as depicted graphically in Figure 9.5.

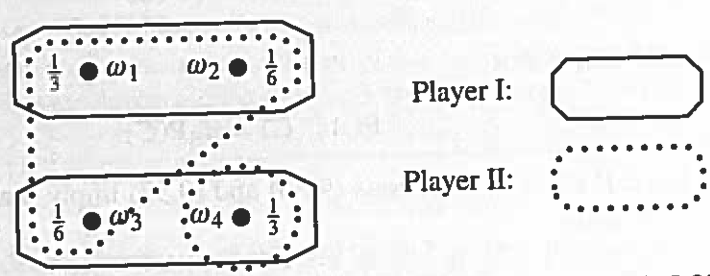


Figure 9.5 The information partitions and the prior distribution in Example 9.33

Let $A = \{\omega_2, \omega_3\}$, and suppose that the true state of the world is ω_3. We will now trace the dynamic process described above. Player I announces the conditional probability $\mathbf{P}(A \mid \{\omega_3, \omega_4\}) = \frac{1}{3}$ that he ascribes to event A, given his information. Notice that in every state of the world Player I ascribes probability $\frac{1}{3}$ to event A, so that this announcement does not add any new information to Player II.

Next, Player II announces the conditional probability $\mathbf{P}(A \mid \{\omega_3, \omega_4\}) = \frac{1}{2}$ that he ascribes to A, given his information. This enables Player I to learn that the true state of the world is not ω_4, because if it were ω_4, Player II would have ascribed conditional probability 0 to the event A.

Player I therefore knows, after Player II's announcement, that the true state of the world is ω_3, and then announces that the conditional probability he ascribes to the event A is 1. This informs Player II that the true state of the world is ω_3, because if the true state of the world were ω_1 or ω_2 (the two other possible states, given Player II's information), Player I would have announced that he ascribed conditional probability $\frac{1}{3}$ to the event A. Player II therefore announces that the conditional probability he ascribes to the event A is 1, and this probability is now common knowledge among the two players.

It is left to the reader to verify that if the true state of the world is ω_1 or ω_2, the dynamic process described above will lead the two players to common knowledge that the conditional probability of the event A is $\frac{1}{3}$. ◀

Aumann's Agreement Theorem has important implications regarding the rationality of betting between two risk-neutral players (or two players who share the same level of risk aversion). To simplify the analysis, suppose that the two players bet that if a certain event A occurs, Player II pays Player I one dollar, and if event A fails to occur, Player I pays Player II one dollar instead. Labeling the probabilities that the players ascribe to event A as q_{I} and q_{II} respectively, Player I should be willing to take this bet if and only if $q_{\mathrm{I}} \geq \frac{1}{2}$, with Player II agreeing to the bet if and only if $q_{\mathrm{II}} \leq \frac{1}{2}$. Suppose that Player I accepts the bet. Then the fact that he has accepted the bet is common knowledge, which means that the fact that $q_{\mathrm{I}} \geq \frac{1}{2}$ is common knowledge. By the same reasoning, if Player II agrees to the bet, that fact is common knowledge, and therefore the fact that $q_{\mathrm{II}} \leq \frac{1}{2}$ is common knowledge. Using a proof very similar to that of Aumann's Agreement Theorem, we conclude that it is impossible for both facts to be common knowledge unless $q_{\mathrm{I}} = q_{\mathrm{II}} = \frac{1}{2}$, in which case the expected payoff for each player is 0, and there is no point in betting (see Exercises 9.29 and 9.30).

Note that the agreement theorem rests on two main assumptions:

- Both players share a common prior over Y.
- The probability that each of the players ascribes to event A is common knowledge among them.

Regarding the first assumption, the common prior distribution $\mathbf{P}$ is part of the Aumann model of incomplete information with beliefs and it is used to compute the players' beliefs given their partitions. As the following example shows, if each player's belief is computed from a different probability distribution, we obtain a more general model in which the agreement theorem does not hold. We will return to Aumann models with incomplete information and different prior distributions in Chapter 10.

Example 9.34 In this example we will show that if the two players have different priors, Theorem 9.32 does not hold. Consider the following Aumann model of incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The information that the two players have is given by

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}. \tag{9.40}$$

- Player I calculates his beliefs based on the following prior distribution:

$$\mathbf{P}_{\mathrm{I}}(\omega_1) = \mathbf{P}_{\mathrm{I}}(\omega_2) = \mathbf{P}_{\mathrm{I}}(\omega_3) = \mathbf{P}_{\mathrm{I}}(\omega_4) = \tfrac{1}{4}. \tag{9.41}$$

- Player II calculates his beliefs based on the following prior distribution:

$$\mathbf{P}_{\mathrm{II}}(\omega_1) = \mathbf{P}_{\mathrm{II}}(\omega_3) = \tfrac{2}{10}, \qquad \mathbf{P}_{\mathrm{II}}(\omega_2) = \mathbf{P}_{\mathrm{II}}(\omega_4) = \tfrac{3}{10}. \tag{9.42}$$

The only connected component in the graph corresponding to this Aumann model is Y (verify!), so that the only event that is common knowledge in any state of the world ω is Y. Let $A = \{\omega_1, \omega_3\}$. A quick calculation reveals that in each state $\omega \in Y$

$$\mathbf{P}_{\mathrm{I}}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{1}{2}, \qquad \mathbf{P}_{\mathrm{II}}(A \mid F_{\mathrm{II}}(\omega)) = \tfrac{2}{5}. \tag{9.43}$$

That is,

$$\left\{\omega : q_{\mathrm{I}} := \mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{1}{2}\right\} = Y, \qquad \left\{\omega : q_{\mathrm{II}} := \mathbf{P}(A \mid F_{\mathrm{II}}(\omega)) = \tfrac{2}{5}\right\} = Y. \tag{9.44}$$

From the definition of the knowledge operator it follows that the facts that $q_{\mathrm{I}} = \frac{1}{2}$ and $q_{\mathrm{II}} = \frac{2}{5}$ are common knowledge in every state of the world. In other words, it is common knowledge in every state of the world that the players ascribe different probabilities to the event A. This does not contradict Theorem 9.32 because the players do not share a common prior. In fact, this result is not surprising; because the players start off by "agreeing" that their initial probability distributions diverge (and that fact is common knowledge), it is no wonder that it is common knowledge among them that they ascribe different probabilities to event A (after learning which partition element they are in). ◀

Example 9.35 In this example we will show that even if the players share a common prior, if the fact that "Player II ascribes probability q_{II} to event A" is not common knowledge, Theorem 9.32 does not hold; that is, it is possible that $q_{\mathrm{I}} \neq q_{\mathrm{II}}$. Consider the following Aumann model of incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The players' information partitions are

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}. \tag{9.45}$$

- The common prior distribution is

$$\mathbf{P}(\omega_1) = \mathbf{P}(\omega_2) = \mathbf{P}(\omega_3) = \mathbf{P}(\omega_4) = \tfrac{1}{4}. \tag{9.46}$$

The partitions $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are depicted graphically in Figure 9.6.

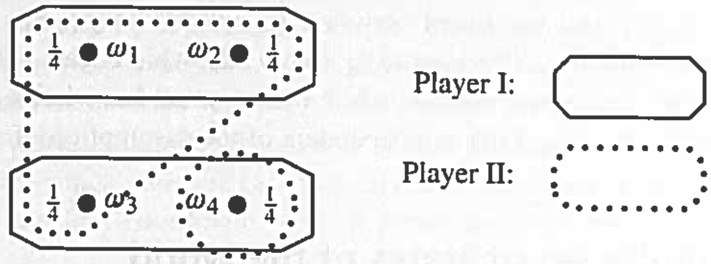


Figure 9.6 The partitions of the players in Example 9.35 and the common prior

The only connected component in the graph corresponding to this Aumann model is Y (verify!). Let $A = \{\omega_1, \omega_3\}$. In each state of the world, the probability that Player I ascribes to event A is $q_{\mathrm{I}} = \frac{1}{2}$:

$$\left\{w \in Y : q_{\mathrm{I}} = \mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{1}{2}\right\} = Y, \tag{9.47}$$

and therefore the fact that $q_{\mathrm{I}} = \frac{1}{2}$ is common knowledge in every state of the world.

In states of the world ω_1, ω_2, and ω_3, Player II ascribes probability $\frac{2}{3}$ to event A:

$$\left\{w \in Y : q_{\mathrm{II}} = \mathbf{P}\left(A \mid F_{\mathrm{II}}(\omega) = \tfrac{2}{3}\right)\right\} = \{\omega_1, \omega_2, \omega_3\} \not\subseteq Y, \tag{9.48}$$

and in state of the world ω_4 he ascribes probability 0 to A. Since the only event that is common knowledge in any state of the world is Y, the event "Player II ascribes probability $\frac{2}{3}$ to A" is not common knowledge in any state of the world. For that reason, the fact that $q_{\mathrm{I}} \neq q_{\mathrm{II}}$ does not contradict Theorem 9.32.

Note that in state of the world ω_1, Player I knows that the state of the world is in $\{\omega_1, \omega_2\}$, and therefore he knows that Player II's information is $\{\omega_1, \omega_2, \omega_3\}$, and thus he (Player I) knows that Player II ascribes probability $q_{\mathrm{II}} = \frac{2}{3}$ to the event A. However, the fact that Player II ascribes probability $q_{\mathrm{II}} = \frac{2}{3}$ to event A is not common knowledge among the players in the state of the world ω_1. This is so because in that state of the world Player II cannot exclude the possibility that the state of the world is ω_3 (he ascribes to this probability $\frac{1}{3}$). If the state of the world is ω_3, Player I knows that the state of the world is in $\{\omega_3, \omega_4\}$, and therefore he (Player I) cannot exclude the possibility that the state of the world is ω_4 (he ascribes to this probability $\frac{1}{2}$), in which case Player II knows that the state of the world is ω_4, and then the probability that Player II ascribes to event A is 0 ($q_{\mathrm{II}} = 0$). Therefore, in state of the world ω_1 Player II ascribes probability $\frac{1}{3}$ to the fact that Player I ascribes probability $\frac{1}{2}$ to Player II ascribing probability 0 to event A. Thus, in state of the world ω_1 Player I knows that $q_{\mathrm{II}} = \frac{2}{3}$, yet this event is not common knowledge among the players. ◀

Before we proceed, let us recall that an Aumann model consists of two elements:

- The partitions of the players, which determine the information (knowledge) they possess.
- The common prior $\mathbf{P}$ that, together with the partitions, determines the beliefs of the players.

The knowledge structure in an Aumann model is independent of the common prior $\mathbf{P}$. Furthermore, as we saw in Example 9.34, even when there is no common prior, and instead every player has a different subjective prior distribution, the underlying knowledge structure and the set of common knowledge events are unchanged. Not surprisingly, the Agreement Theorem (Theorem 9.32), which deals with beliefs, depends on the assumption of a common prior, while the common knowledge characterization theorem (Theorem 9.24, page 333) is independent of the assumption of a common prior.

9.3 An infinite set of states of the world

Thus far in the chapter, we have assumed that the set of states of the world is finite. What if this set is infinite? With regard to set-theoretic operations, in the case of an infinite set of states of the world we can make use of the same operations that we implemented in the finite case. On the other hand, dealing with the beliefs of the players requires using tools from probability theory, which in the case of an infinite set of states of the world means that we need to ensure that this set is a measurable space.

A *measurable space* is a pair $(Y, \mathcal{F})$, with Y denoting a set, and $\mathcal{F}$ a σ-algebra over Y. This means that $\mathcal{F}$ is a family of subsets of Y that includes the empty set, is closed under complementation (i.e., if $A \in \mathcal{F}$ then $A^c = Y \setminus A \in \mathcal{F}$), and is closed under countable unions (i.e., if $(A_n)_{n=1}^{\infty}$ is a family of sets in $\mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$). An event is any element of $\mathcal{F}$. In particular, the partitions of the players, $\mathcal{F}_i$, are composed solely of elements of $\mathcal{F}$.

The collection of all the subsets of Y, 2^Y, is a σ-algebra over Y, and therefore $(Y, 2^Y)$ is a measurable space. This is in fact the measurable space we used, without specifically mentioning it, in all the examples we have seen so far in which Y was a finite set. All the infinite sets of states of the world Y that we will consider in the rest of the section will be a subset of a Euclidean space, and the σ-algebra $\mathcal{F}$ will be the σ-algebra of Borel sets, that is, the smallest σ-algebra that contains all the relatively open sets[11] in Y.

The next example shows that when the set of states of the world is infinite, knowledge is not equivalent to belief with probability 1 (in contrast to the finite case; see Theorem 9.29 on page 336).

Example 9.36 Consider an Aumann model of incomplete information in which the set of players $N = \{\mathrm{I}\}$ contains only one player, the set of states of the world is $Y = [0, 1]$, the σ-algebra $\mathcal{F}$ is the σ-algebra of Borel sets,[12] and the player has no information, which means that $\mathcal{F}_{\mathrm{I}} = \{Y\}$. The common prior $\mathbf{P}$ is the uniform distribution over the interval $[0, 1]$.

Since there is only one player and his partition contains only one element, the only event that the player knows (in any state of the world ω) is Y. Let A be the set of irrational numbers in the interval $[0, 1]$, which is in $\mathcal{F}$. As the set A does not contain Y, the player does not know A. But $\mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \mathbf{P}(A \mid Y) = \mathbf{P}(A) = 1$ for all $\omega \in Y$. ◀

11 When $Y \subseteq \mathbb{R}^d$, a set $A \subseteq Y$ is *relatively open in* Y if it is equal to the intersection of Y with an open set in $\mathbb{R}^d$.

12 In this case the σ-algebra of Borel sets is the smallest σ-algebra that contains all the open intervals in $[0, 1]$, and the intervals of the form $[0, \alpha)$ and $(\alpha, 1]$ for $\alpha \in (0, 1)$.

Next we show that when the set of states of the world is infinite, the very notion of knowledge hierarchy can be problematic. To make use of the knowledge structure, for every event $A \in \mathcal{F}$ the event $K_i A$ must also be an element of $\mathcal{F}$: if we can talk about the event A, we should also be able to talk about the event that "player i knows A."

Is it true that for every σ-algebra, every partition $(\mathcal{F}_i)_{i \in N}$ representing the information of the players, and every event $A \in \mathcal{F}$, it is necessarily true that $K_i A \in \mathcal{F}$? When the set of states of the world is infinite, the answer to that question is no. This is illustrated in the next example, which uses the fact that there is a Borel set in the unit square whose projection onto the first coordinate is not a Borel set in the interval [0, 1] (see Suslin [1917]).

Example 9.37 Consider the following Aumann model of incomplete information:

- There are two players: $N = \{\mathrm{I}, \mathrm{II}\}$.
- The space of states of the world is the unit square: $Y = [0, 1] \times [0, 1]$, and $\mathcal{F}$ is the σ-algebra of Borel sets in the unit square.
- For $i = \mathrm{I}, \mathrm{II}$, the information of player i is the i-th coordinate of ω; that is, for each $x, y \in [0, 1]$ denote

$$A_x = \{(x, y) \in Y : 0 \leq y \leq 1\}, \qquad B_y = \{(x, y) \in Y : 0 \leq x \leq 1\}. \tag{9.49}$$

A_x is the set of all points in Y whose first coordinate is x, and B_y is the set of all points in Y whose second coordinate is y. We then have

$$\mathcal{F}_{\mathrm{I}} = \{A_x : 0 \leq x \leq 1\}, \qquad \mathcal{F}_{\mathrm{II}} = \{B_y : 0 \leq y \leq 1\}. \tag{9.50}$$

In words, Player I's partition is the set of vertical sections of Y, and the partition of Player II is the set of horizontal sections of Y. Thus, for any $(x, y) \in Y$ Player I knows the x-coordinate and Player II knows the y-coordinate.

Let $E \subseteq Y$ be a Borel set whose projection onto the x-axis is not a Borel set, i.e., the set

$$F = \{x \in [0, 1] : \text{there exists } y \in [0, 1] \text{ such that } (x, y) \in E\} \tag{9.51}$$

is not a Borel set, and hence $F^c = Y \setminus F$ is also not a Borel set in [0, 1]. Player I knows that the event E does not obtain when the x-coordinate is not in F:

$$K_{\mathrm{I}}(E^c) = F^c \times [0, 1]. \tag{9.52}$$

This implies that despite the fact that the set E^c is a Borel set, the set of states of the world in which Player I knows the event E^c is not a Borel set. ◀

In spite of the technical difficulties indicated by Examples 9.36 and 9.37, in Chapter 10 we develop a general model of incomplete information that allows infinite sets of states of the world.

9.4 The Harsanyi model of games with incomplete information

In our treatment of the Aumann model of incomplete information, we concentrated on concepts such as mutual knowledge and mutual beliefs among players regarding the true