Group Decisions for Two Candidates

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Notation

- $X = \{a, b\}$ is the set of candidates and $N = \{1, \ldots, n\}$ is the set of voters.
- There are three possible preference relations on X: $O(X) = \{R_1, R_2, R_3\}$, where $R_1 = \{(a, b)\}$, $R_2 = \{(b, a)\}$ and $R_3 = \{(a, b), (b, a)\}$.

  Simplifying notation: To simplify the notation, let

  $\mathsf{1}$ denote the relation R_1 (in which a is ranked above b),

  $-\mathsf{1}$ is the relations R_2 (in which b is ranked above a) and

  $\mathsf{0}$ is the relation R_3 (in which a and b are tied).

- Recall that if X is a set, then X^n is the n-fold cross product of X with itself. For instance,

$$\{\mathsf{1}, -\mathsf{1}, \mathsf{0}\}^2 = \{(\mathsf{1}, \mathsf{1}), (\mathsf{1}, -\mathsf{1}), (\mathsf{1}, \mathsf{0}), (-\mathsf{1}, -\mathsf{1}), (-\mathsf{1}, \mathsf{1}), (-\mathsf{1}, \mathsf{0}), (\mathsf{0}, -\mathsf{1}), (\mathsf{0}, \mathsf{1}), (\mathsf{0}, \mathsf{0})\}$$

- A group decision method for V is a function $F : O(X)^n \to O(X)$. Given, our simplifying notation, a group decision method is a function

$$F : \{\mathsf{1}, \mathsf{0}, -\mathsf{1}\}^n \to \{\mathsf{1}, \mathsf{0}, -\mathsf{1}\}$$

- Suppose that $v \in \{\mathsf{1}, \mathsf{0}, -\mathsf{1}\}$. Then, define $-v$ as follows:

$$-v = \begin{cases} \mathsf{1} & \text{if } v = -\mathsf{1} \\ \mathsf{0} & \text{if } v = \mathsf{0} \\ -\mathsf{1} & \text{if } v = \mathsf{1} \end{cases}$$

  For a profile $\mathbf{R} = (v_1, v_2, \ldots, v_n) \in \{\mathsf{1}, \mathsf{0}, -\mathsf{1}\}^n$, let $-\mathbf{R} = (-v_1, -v_2, \ldots, -v_n)$.

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- A **permutation** of the set of agents N is a 1-1 function $\pi : N \to N$. For instance, if $N = \{1, 2, 3\}$, then $\pi : N \to N$ where $\pi(1) = 3$, $\pi(3) = 1$, and $\pi(2) = 2$ is a permutation. For a profile $\mathbf{R} = (v_1, v_2, \ldots, v_n)$, let $\pi(\mathbf{R}) = (v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$. That is, $\pi(\mathbf{R})$ is a "shuffling" of the profile. For example, if $\mathbf{R} = (1, 0, -1)$ and π is the above permutation, then $\pi(\mathbf{R}) = (-1, 0, 1)$.

- For a profile $\mathbf{R} = (R_1, \ldots, R_n)$ and $v \in \{1, 0, -1\}$, let $N_{\mathbf{R}}(v) = |\{i \mid R_i = v\}|$. For instance, if $\mathbf{R} = (1, -1, 0, 0, 1, 1, -1, -1, 1)$. Then, $N_{\mathbf{R}}(1) = 4$, $N_{\mathbf{R}}(0) = 2$, and $N_{\mathbf{R}}(-1) = 3$

- For each profile $\mathbf{R}$ and $i \in V$, let $\mathbf{R}_{-i}$ be the profile of all rankings in $\mathbf{R}$ except for voter i's ranking. For instance, if $\mathbf{R} = (1, 0, -1, 1)$, then $\mathbf{R}_{-3} = (1, 0, 1)$.

- Suppose that $\mathbf{R} = (v_1, \ldots, v_n)$ and $\mathbf{R}' = (v'_1, \ldots, v'_n)$. We say that $\mathbf{R}'$ is a monotonic improvement of $\mathbf{R}$ for a provided there is a voter $i \in V$ such that $v_i = -1$ and $v'_i = 1$ or $v'_i = 0$ and for all $j \neq i$, $v'_j = v_j$. Similarly, we say that $\mathbf{R}'$ is a monotonic improvement of $\mathbf{R}$ for b provided there is a voter $i \in V$ such that $v_i = 1$ and $v'_i = -1$ or $v'_i = 0$ and for all $j \neq i$, $v'_j = v_j$. For instance $(1, 0, -1)$ is a monotonic improvement of $(1, -1, -1)$ for a, $(-1, 0, -1)$ is a monotonic improvement of $(1, 0, -1)$ for b, but $(0, 0, -1)$ is not a monotonic improvement of $(1, 0, -1)$ for a.

Group decision methods

Dictator Choose a voter $i \in V$. For each $\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots, R_n) \in \{1, 0, -1\}^n$, let $F_i(\mathbf{R}) = R_i$.

Imposed rule There are two group decision methods, one for candidate a (F_a) and one for b (F_b): For each $\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots, R_n) \in \{1, 0, -1\}$, let $F_a(\mathbf{R}) = R_1$ and let $F_b(\mathbf{R}) = R_2$.

Ignoring a voter Choose a voter $i \in V$. For each $\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots, R_n) \in \{1, 0, -1\}^n$, let

$$F_{-i}(\mathbf{R}) = \begin{cases} 1 & \text{if } N_{\mathbf{R}_{-i}}(1) > N_{\mathbf{R}_{-i}}(-1) \\ 0 & \text{if } N_{\mathbf{R}_{-i}}(1) = N_{\mathbf{R}_{-i}}(-1) \\ -1 & \text{if } N_{\mathbf{R}_{-i}}(1) < N_{\mathbf{R}_{-i}}(-1) \end{cases}$$

Ballot stuffing Give a candidates two extra votes. There are two group decision methods, one for a (F_{a+2}) and one for b (F_{b+2}). For each $\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots, R_n) \in \{1, 0, -1\}^n$, let

$$F_{a+2}(\mathbf{R}) = \begin{cases} 1 & \text{if } N_{\mathbf{R}}(1) + 2 > N_{\mathbf{R}}(-1) \\ 0 & \text{if } N_{\mathbf{R}}(1) + 2 = N_{\mathbf{R}}(-1) \\ -1 & \text{if } N_{\mathbf{R}}(1) + 2 < N_{\mathbf{R}}(-1) \end{cases}$$

and

$$F_{b+2}(\mathbf{R}) = \begin{cases} 1 & \text{if } N_{\mathbf{R}}(1) > N_{\mathbf{R}}(-1) + 2 \\ 0 & \text{if } N_{\mathbf{R}}(1) = N_{\mathbf{R}}(-1) + 2 \\ -1 & \text{if } N_{\mathbf{R}}(1) < N_{\mathbf{R}}(-1) + 2 \end{cases}$$

Reversal Swap the a and b in each ranking, then selection the candidate(s) with the most first place votes. For each $\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots, R_n) \in \{1, 0, -1\}^n$, let

$$F_{rev}(\mathbf{R}) = \begin{cases} 1 & \text{if } N_{-\mathbf{R}}(1) > N_{-\mathbf{R}}(-1) \\ 0 & \text{if } N_{-\mathbf{R}}(1) = N_{-\mathbf{R}}(-1) \\ -1 & \text{if } N_{-\mathbf{R}}(1) < N_{-\mathbf{R}}(-1) \end{cases}$$

Majority Rule The candidate with the most first place votes wins. For each $\mathbf{R} = (R_1, R_2, \ldots, R_i, \ldots, R_n)$ $\{1, 0, -1\}^n$, let

$$F_{maj}(\mathbf{R}) = \begin{cases} 1 & \text{if } N_{\mathbf{R}}(1) > N_{\mathbf{R}}(-1) \\ 0 & \text{if } N_{\mathbf{R}}(1) = N_{\mathbf{R}}(-1) \\ -1 & \text{if } N_{\mathbf{R}}(1) < N_{\mathbf{R}}(-1) \end{cases}$$

Postulates

Decisiveness Every input to the group decision method must be associated with a single group decision. For each profile $\mathbf{R}$, there is a unique R such that R is the group decision associated with $\mathbf{R}$.

Anonymity All voters should be be treated equally (The shuffling of a profile does not affect the winner). For all $\mathbf{R} \in \{1, 0, -1\}^n$ and permutations $\pi : N \to N$, $F(\mathbf{v}) = F(\pi(\mathbf{v}))$

Neutrality All candidates should be treated equally: For all $\mathbf{R} \in \{1, 0, -1\}^n$, $F(-\mathbf{R}) = -F(\mathbf{R})$

Monotonicity For all $\mathbf{R}, \mathbf{R}'$, if $\mathbf{R}$ is a monotonic improvement of $\mathbf{R}'$ for a, then if $F(\mathbf{R}') = 1$ or $F(\mathbf{R}') = 0$, then $F(\mathbf{R}) = 1$. Similarly, if $\mathbf{R}$ is a monotonic improvement of $\mathbf{R}'$ for b, then if $F(\mathbf{R}') = -1$ or $F(\mathbf{R}') = 0$, then $F(\mathbf{R}) = -1$.

Fill in the table

	Decisiveness	Anonymity	Neutrality	Monotonicity
Dictator				
Imposed Rule				
Ignoring				
Ballot Stuffing				
Reversal				
Majority				

May's Theorem. A group decision method satisfies decisiveness, anonymity, neutrality and monotonicity if, and only if, the group decision method is majority rule.

Question: Majority rule satisfies unanimity. Why don't we need to list unanimity as postulate in May's Theorem?