PHIL309P
Philosophy, Politics and Economics

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Announcements

- Course website
  https://myelms.umd.edu/courses/1133211
- Reading: Gaus, Ch 2. (up to 2.3) Utility Theory;Reiss, Ch 3, pgs. 29 - 42; Gilboa dialogue.
- Weekly writing: **Due Wednesday, 11.59pm**.
- Office hours canceled this Wednesday.

Preferences

Preferring or choosing x is different that "liking" x or "having a taste for x": one can prefer x to y but *dislike* both options

In utility theory, preferences are always understood as comparative: "preference" is more like "bigger" than "big"

Representing Preferences

A relation $\succeq \subseteq X \times X$ is a **(rational) preference relation** (for a decision maker) provided

1. $\succeq$ is complete (and hence reflexive)
2. $\succeq$ is transitive

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Suppose that $\succeq$ is a preference relation. Then,

- **Strict preference**: $x \succ y$ iff $x \succeq y$ and $y \not\succeq x$
- **Indifference**: $x \sim y$ iff $x \succeq y$ and $y \succeq x$

- What is the relationship between choice and preference?
- Why *should* preferences be complete and transitive?
- *Are* people's preferences complete and transitive?

Folk Psychology

The view that human behavior can and ought to be explained by citing beliefs and desires.

Beliefs and desires are thus *reasons for action*.

No every reason an individual might have to perform an action also constitute the reason that explains his or her action. Rather it is the reason the individual *acted on* that explains the action.

Folk Psychology

In order to infer motivations or beliefs from behavior (or other accessible forms of evidence), one must make fairly strong assumptions concerning the system of beliefs and desires people have. If individuals acted very erratically (though always on reasons!) it would be impossible to infer beliefs or desires or both both from their actions.

Choices

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Decisions are between beliefs and desires on the one hand and actions on the other.

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> The verb "to prefer" can either mean "to choose" or "to like better," and these two senses are frequently confused in economic literature. That fact that an individual chooses A rather than B is far from conclusive evidence that he likes A better. But whether he likes A better or not should be completely irrelevant to the theory of price. (Little, 1949).

Preferences and Choices

Preferences are closely related to choices: preferences may *cause* and help to *explain* choices; preferences may be invoked to *justify* choices, in fortuitous circumstances, we can use preference data to make *predictions* about choice. **But to identify the two would be a mistake**.

Preferences and Choices

- We have preferences over vastly more states of affairs than we can ever hope (or dread) to be in the position to choose.

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Preferences and Choices

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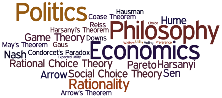
- What about *counter-preferential choice*?
- Preferences must be *stable* over a reasonable amount of time in a way that (observed) choices aren't (needed to predict and explain choices).
- Beliefs and expectations over future states of affairs are needed in addition to preferences in order to explain choices. To banish preferences understood as mental rankings because they are unobservable or subjective would mean that beliefs and expectations would have to be banished as well.

Preferences will be understood as *mental rankings* of alternatives “all things considered”.

Revealed Preference Theory



Standard economics focuses on revealed preference because economic data comes in this form. Economic data can—at best—reveal what the agent wants (or has chosen) in a particular situation. Such data do not enable the economist to distinguish between what the agent intended to choose and what he ended up choosing; what he chose and what he ought to have chosen. (Gul and Pesendorfer, 2008)

Sen's α Condition

R: red wine

W: white wine

L: lemonade

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Sen's α Condition

R: red wine W: white wine L: lemonade	R: red wine W: white wine

If the world champion is American, then she must be a US champion too.

Observations of actual choices will only partially constrain preference attribution. That someone chooses red wine when white wine is available does not allow one to conclude that the choice of an white wine was ruled out by her preferences, only that her preferences ruled the red wine in.

Sen's β Condition

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Sen's β Condition

R: red wine

W: white wine

L: lemonade

Sen's β Condition

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Sen's β Condition

R: red wine

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R: red wine

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If some American is a world champion, then all champions of America must be world champions.

Revealed Preference Theory

A decision maker's choices over a set of alternatives X are **rationalizable** iff there is a (rational) preference relation on X such that the decision maker's choices *maximize* the preference relation.

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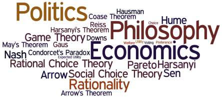

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Revelation Theorem. A decision maker's choices satisfy Sen's α and β if and only if the decision maker's choices are **rationalizable**.

Choice Functions

Suppose X is a set of options. And consider $B \subseteq X$ as a choice problem. A **choice function** is any function where $C(B) \subseteq B$. B is sometimes called a menu and $C(B)$ the set of "rational" or "desired" choices.

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Maximizing

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“The formulation of maximizing behavior in economics has often parallels the modeling of maximization in physics an related disciplines. But maximizing *behavior* differs from nonvolitional maximization because of the fundamental relevance of the choice act, which has to be placed in a central position in analyzing maximizing behavior. A person’s preferences over *comprehensive* outcomes (including the choice process) have to be distinguished form the conditional preferences over *culmination* outcomes *given* the act of choice.” (pg. 745)

Maximizing

You arrive at a garden party and can readily identify the most comfortable chair. You would be delighted if an imperious host were to assign you that chair. However, if the matter is left to your own choice, you may refuse to rush to it.

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Invoking someone’s preferences will suffice to explain why some choices were not made (i.e. in terms of rational impermissibility) but not typically why some particular choice was made. To take up the slack, explanations must draw on factors other than preference: psychological one such as the framing of the choice problem or the saliency of particular options, or sociological ones such as the existence of norms or conventions governing choices of the relevant kind.

Ordinal Utility Theory

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What properties does such a preference ordering have?

Ordinal Utility Theory

Fact. Suppose that X is finite and $\succeq$ is a complete and transitive ordering over X, then there is a utility function $u : X \to \Re$ that represents $\succeq$

(i.e., $x \succeq y$ iff $u(x) \geq u(y)$)

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Fact. Suppose that X is finite and $\succeq$ is a complete and transitive ordering over X, then there is a utility function $u : X \to \Re$ that represents $\succeq$

(i.e., $x \succeq y$ iff $u(x) \geq u(y)$)

Utility is *defined* in terms of preference (so it is an error to say that the agent prefers x to y *because* she assigns a higher utility to x than to y).

Important

All three of the utility functions represent the preference $x \succ y \succ z$

Item	u_1	u_2	u_3
x	3	10	1000
y	2	5	99
z	1	0	1

$x \succ y \succ z$ is represented by both $(3, 2, 1)$ and $(1000, 999, 1)$, so one cannot say that y is "closer" to x than to z.

$$X = \{M, C, P, L\}$$

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$M\ C\ P\ L$	$M\ C$	$P\ L$
$M\ P\ L$	$M\ P$	M
$C\ P\ L$	$M\ L$	C
$M\ C\ P$	$C\ P$	P
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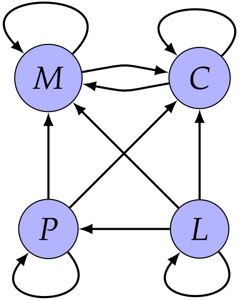

$$\succeq = \{(M, C), (C, M), (M, P), (M, L), (C, P), (C, L), (P, L),$$
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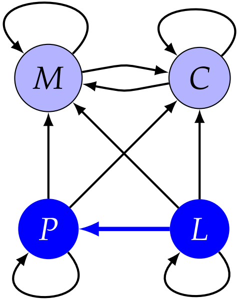

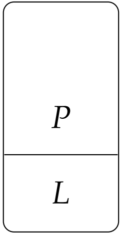

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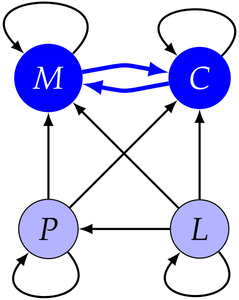

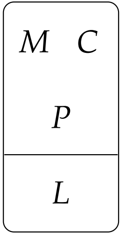

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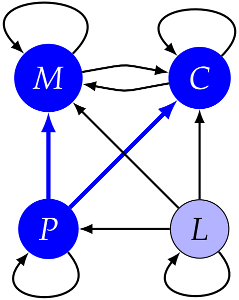

$M \quad C$
P
L

$$\succeq = \{(M,C),(C,M),(M,P),(M,L),(C,P),(C,L),(P,L),\\ (M,M),(P,P),(C,C),(L,L)\}$$

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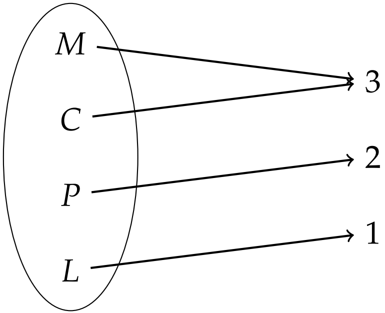

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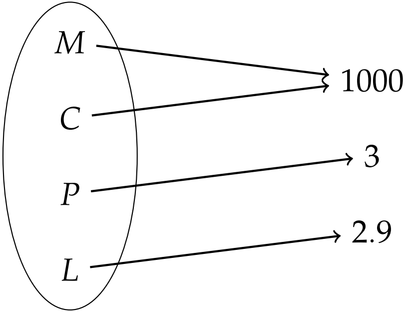

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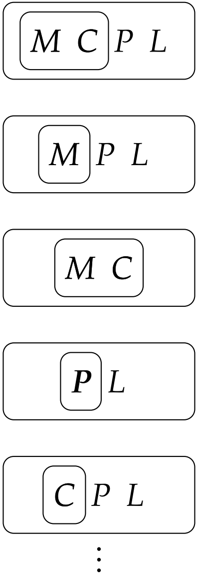
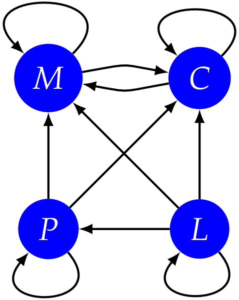
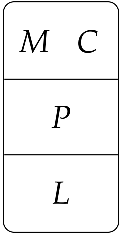
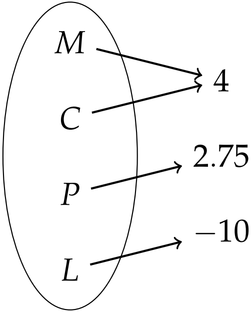


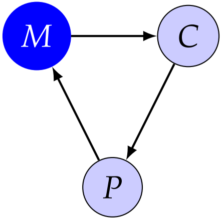




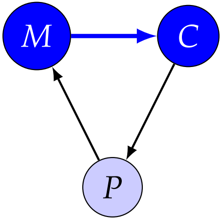


- What is the relationship between choice and preference?
- Why *should* preferences be complete and transitive?
- *Are* people's preferences complete and transitive?

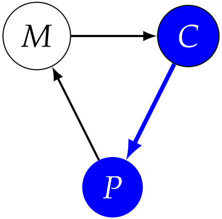
- Transitivity: Money-pump argument
- Completeness: Incommensurable options



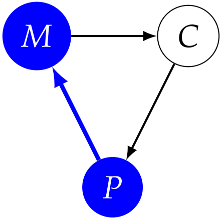
(M)



$$(M) \implies (C, -1)$$



$$(M) \implies (C, -1) \implies (P, -2)$$



$$(M) \implies (C, -1) \implies (P, -2) \implies (M, -3)$$

$$
\begin{tikzcd}
M \arrow[rr] & & C \arrow[dl] \\
& P \arrow[ul] &
\end{tikzcd}
$$

$$(M) \implies (C, -1) \implies (P, -2) \implies (M, -3) \implies (C, -4) \implies \cdots$$

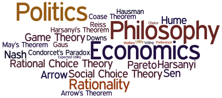
[O]f all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others, it is inaccurate as a description of real life; but unlike them we find it hard to accept even from the normative viewpoint.

(Aumann, 1962)

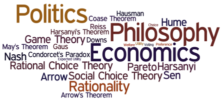
Rather than trying to provide instrumental or pragmatic justifications for the axioms of ordinal utility, it is better...to see them as constitutive of our conception of a fully rational agent....those disposed to blatantly ignore transitivity are unintelligible to us: we can't understand their pattern of actions as sensible.

[G], pg. 39

Preference, Choice, and Utility



- ✓ Representing *preferences*: relations, preference axioms
- ✓ *Revealed* preference theory: WARP, Sen's α and β, Revelation Theorem
- ▸ *Utility*: Ordinal vs. cardinal utility, interval scale, ratio scale
- ▸ *Expected utility theory*: (probability), von Neumann-Morgenstern Theorem, Allais paradox, Ellsberg paradox, (Other issues: framing effects, state-dependent utility, etc.)
- ▸ Interpersonal comparison of utilities



- Reading: Gaus, Ch 2;Reiss, Ch 3; Briggs, Normative Expected Utility Theory.
- Mathematical background: my notes on choice, preference and utility.
- Weekly writing: **Due Wednesday, 11.59pm**.

Cardinal Utility Theory

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Key idea: Ordinal preferences over *lotteries* allows us to infer a cardinal scale (with some additional axioms).

John von Neumann and Oskar Morgenstern. *The Theory of Games and Economic Behavior*. Princeton University Press, 1944.

Axioms of Cardinal Utility

Suppose that X is a set of outcomes and consider **lotteries over** X (i.e., probability distributions over X)

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A **compound lottery** is $\alpha L + (1 - \alpha)L'$ meaning "play lottery L with probability α and L' with probability $1 - \alpha$"

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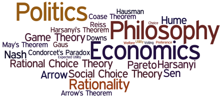

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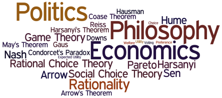
A **compound lottery** is $\alpha L + (1 - \alpha)L'$ meaning "play lottery L with probability α and L' with probability $1 - \alpha$"

Running example: Suppose Ann prefers pizza (p) over taco (t) over yogurt (y) ($p \succ t \succ y$) and consider the different lotteries where the prizes are p, t and y.

Continuity



Continuity: for all options x, y and z if $x \succeq y \succeq z$, there is some lottery L with probability p of getting x and $(1-p)$ of getting y such that the agent is indifferent between L and y.

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Consider the lottery $L = 0.99$ get y and 0.01 get p

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Would Ann trade t for L?

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Suppose Ann has t.

Consider the lottery $L = 0.99$ get y and 0.01 get p
Would Ann trade t for L?

Consider the lottery $L' = 0.99$ get p and 0.01 get y
Would Ann trade t for L'?

Continuity says that there is must be some lottery where Ann is indifferent between keeping t and playing the lottery.

Better Prizes

Better Prizes: suppose L_1 is a lottery over (w, x) and L_2 is over (y, z) suppose that L_1 and L_2 have the same probability over prizes. The lotteries each have an equal prize in one position they have unequal prizes in the other position then if L_1 is the lottery with the better prize then $L_1 \succ L_2$; if neither lottery has a better prize then $L_1 \approx L_2$.

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Since Ann prefers p to t, this axiom says that Ann prefers L_1 to L_2

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This eliminates utility from the thrill of gambling and so the only ultimate concern is the prizes.

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- Deep issues about how to identify correct descriptions of the outcomes and options.

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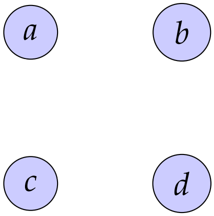

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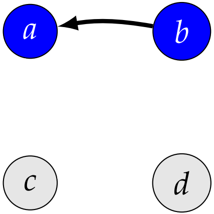

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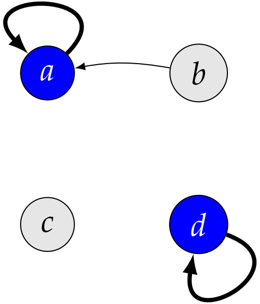

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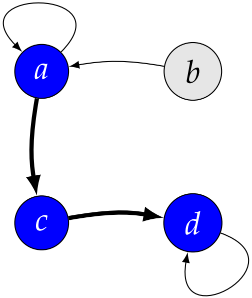

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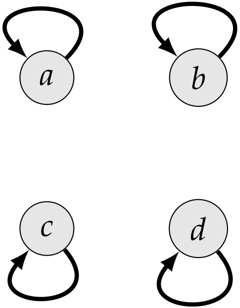

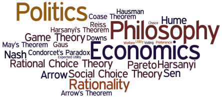

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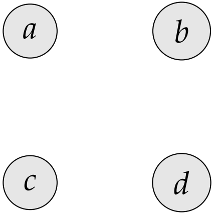

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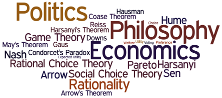

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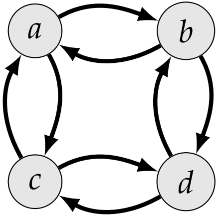

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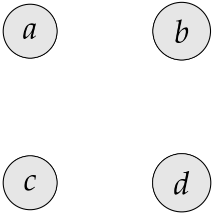

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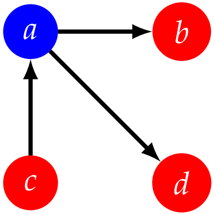

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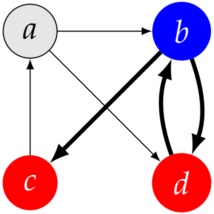

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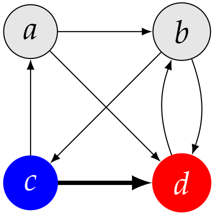

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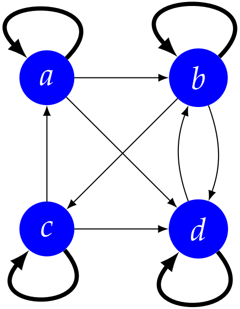

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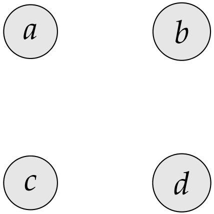

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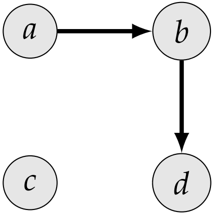

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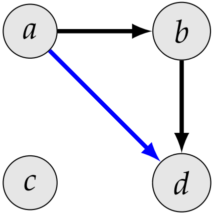

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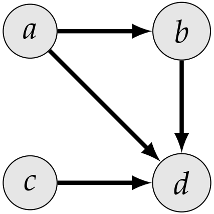

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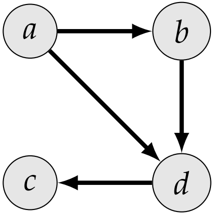

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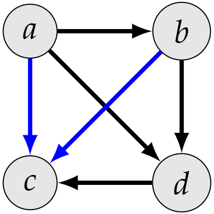

Maximal elements, Cycles

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$x \in X$ is **maximal** with respect to R provided there is no $y \in X$ such that $y\ R\ x$.

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A **cycle** is a set of distinct elements $x_1, \ldots, x_n$ such that

$$x_1\ R\ x_2 \cdots x_{n-1}\ R\ x_n\ R\ x_1$$

R is **acyclic** if it does not contain any cycles.

Representing Preferences

Let X be a set of options/outcomes. A decision maker's *preference* over X is represented by a *relation* $\succeq \subseteq X \times X$.

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