PHIL309P
Philosophy, Politics and Economics

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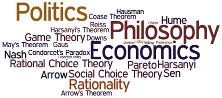

Announcements

- Course website
  https://myelms.umd.edu/courses/1133211
- Reading
  - Gaus, Ch. 5
  - EP, Voting Methods (Stanford Encyclopedia of Philosophy)
  - C. List, Social Choice Theory (Stanford Encyclopedia of Philosophy)
  - M. Morreau, Arrow's Theorem (Stanford Encyclopedia of Philosophy)
- Quiz

$$F : L(X)^n \to (\wp(X) - \emptyset)$$

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Liberalism: For all voters $i \in N$, there exists two alternatives A_i and B_i such that for all profiles $\mathbf{R} \in L(X)^n$, if $A_i \ R_i \ B_i$, then $B \notin F(\mathbf{R})$. That is, i is **decisive** over A_i and B_i.

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Minimal Liberalism: There are two distinct voters i and j such that there are alternatives A_i, B_i, A_j, and B_j such that i is decisive over A_i and B_i and j is decisive over A_j and B_j.

Sen's Impossibility Theorem. Suppose that X contains at least three elements. No social choice function $F : L(X)^n \rightarrow (\wp(X) - \emptyset)$ satisfies (universal domain) and both minimal liberalism and the Pareto condition.

A. Sen. *The Impossibility of a Paretian Liberal*. Journal of Political Economy, 78:1, pp. 152 - 157, 1970.

Suppose that X contains at least three elements and there are elements A, B, C and D such that

1. Voter 1 is decisive over A and B: for any profile $\mathbf{R} \in L(X)^n$, if $A \ R_1 \ B$, then $B \notin F(\mathbf{R})$
2. Voter 2 is decisive over C and D: for any profile $\mathbf{R} \in L(X)^n$, if $C \ R_2 \ D$, then $D \notin F(\mathbf{R})$

Two cases: 1. $B \neq C$ and 2. $B = C$.

Suppose that $X = \{A, B, C, D\}$ and

- Voter 1 is decisive over the pair A, B
- Voter 2 is decisive over the pair C, D

1	2
D	B
A	C
B	D
C	A

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for C, D implies $D \notin F(\mathbf{R})$

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for C, D implies $D \notin F(\mathbf{R})$

Pareto implies $A \notin F(\mathbf{R})$

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for C, D implies $D \notin F(\mathbf{R})$

Pareto implies $A \notin F(\mathbf{R})$

Pareto implies $C \notin F(\mathbf{R})$

Suppose that $X = \{A, B, C\}$ and

- Voter 1 is decisive over the pair A, B
- Voter 2 is decisive over the pair B, C
- Voter 1's preference $R_1 \in L(X)$ is $C \; R_1 \; A \; R_1 \; B$
- Voter 2's preference $R_2 \in L(X)$ is $B \; R_2 \; C \; R_2 \; A$

1	2
C	B
A	C
B	A

1	2
C	B
A	C
B	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

1	2
C	B
A	C
B	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for B, C implies $C \notin F(\mathbf{R})$

1	2
C	B
A	C
B	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for B, C implies $C \notin F(\mathbf{R})$

Pareto implies $A \notin F(\mathbf{R})$

“What is the moral?

"What is the moral? It is that in a very basic sense liberal values conflict with the Pareto principle. If someone takes the Pareto principle seriously, as economists seem to do, then he has to face problems of consistency in cherishing liberal values, even very mild ones....

“What is the moral? It is that in a very basic sense liberal values conflict with the Pareto principle. If someone takes the Pareto principle seriously, as economists seem to do, then he has to face problems of consistency in cherishing liberal values, even very mild ones.... While the Pareto criterion has been thought to be an expression of individual liberty, it appears that in choices involving more than two alternatives it can have consequences that are, in fact, deeply illiberal.” (pg. 157)

A. Sen. *The Impossibility of a Paretian Liberal*. Journal of Political Economy, 78:1, pp. 152 - 157, 1970.

Re-examining the the social choice problem: Maximizing *social welfare*

Social Utility?

Utilitarianism (Bentham, Mill, etc.): Place at the top the social options that produce the *greatest amount of pleasure for the citizenry as a whole*

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How are we to *measure* the amount of pleasure available under each social option?

A reminder on modern utility theory...

Utility Function

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

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A preference ordering is **represented** by a utility function iff x is (weakly) preferred to y provided $u(x) \geq u(y)$

What properties does such a preference ordering have?

$$X = \{M, C, P, L\}$$

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$M\ C\ P\ L$	$M\ C$	$P\ L$
$M\ P\ L$	$M\ P$	M
$C\ P\ L$	$M\ L$	C
$M\ C\ P$	$C\ P$	P
$M\ C\ L$	$C\ L$	L

$$X = \{M, C, P, L\}$$

$\boxed{M\ C}\ P\ L$	$\boxed{M\ C}$	$\boxed{P}\ L$
$\boxed{M}\ P\ L$	$\boxed{M}\ P$	$\boxed{M}$
$\boxed{C}\ P\ L$	$\boxed{M}\ L$	$\boxed{C}$
$\boxed{M\ C}\ P$	$\boxed{C}\ P$	$\boxed{P}$
$\boxed{M\ C}\ L$	$\boxed{C}\ L$	$\boxed{L}$

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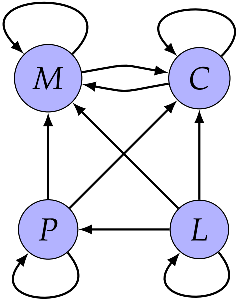

$$\succeq = \{(M, C), (C, M), (M, P), (M, L), (C, P), (C, L), (P, L),$$
$$(M, M), (P, P), (C, C), (L, L)\}$$

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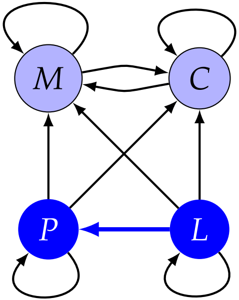

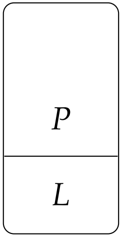

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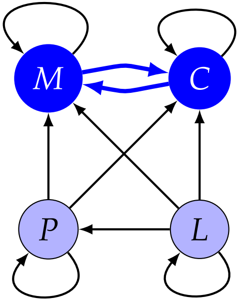

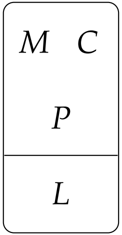

$$\succeq = \{(M,C),(C,M),(M,P),(M,L),(C,P),(C,L),(P,L),\\ (M,M),(P,P),(C,C),(L,L)\}$$

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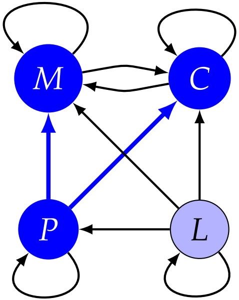

$M \quad C$
P
L

$$\succeq = \{(M, C), (C, M), (M, P), (M, L), (C, P), (C, L), (P, L), (M, M), (P, P), (C, C), (L, L)\}$$

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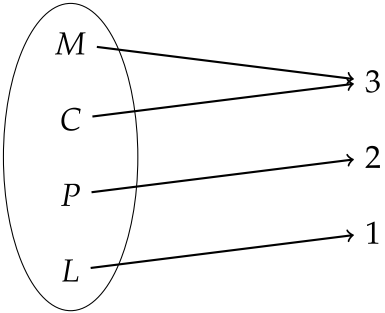

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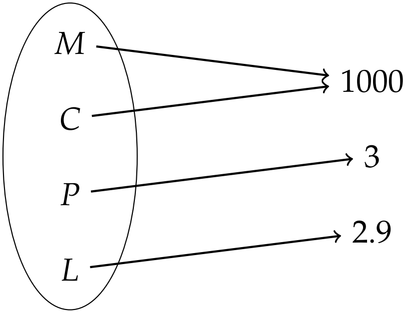

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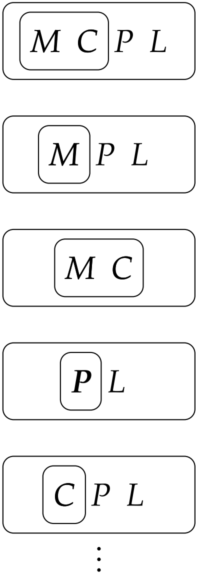

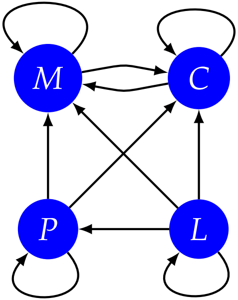

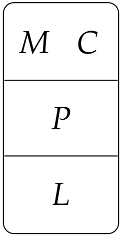

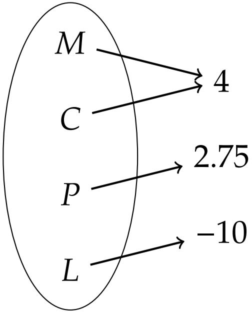

Important

All three of the utility functions represent the preference $x \succ y \succ z$

Item	u_1	u_2	u_3
x	3	10	1000
y	2	5	99
z	1	0	1

$x \succ y \succ z$ is represented by both $(3, 2, 1)$ and $(1000, 999, 1)$, so one cannot say that y is "closer" to x than to z.

Ordinal vs. Cardinal Utility

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> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
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> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb is twice as much as 5lb. But, 10kg is not twice as much as 5lb.

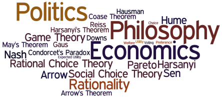
Suppose that X is a set of outcomes.

A **(simple) lottery** over X is denoted $[x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$ where for $i = 1, \ldots, n$, $x_i \in X$ and $p_i \in [0, 1]$, and $\sum_i p_i = 1$.

Let $\mathcal{L}$ be the set of (simple) lotteries over X. We identify elements $x \in X$ with the lottery $[x : 1]$.

Suppose that $\succeq$ is a relation on $\mathcal{L}$.

Axioms



Preference	$\succeq$ is reflexive, transitive and complete
Compound Lotteries	The decision maker is indifferent between every compound lottery and the *corresponding* simple lottery.
Independence	For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0,1]$, $L_1 \succ L_2$ if, and only if, $[L_1 : a, L_3 : (1-a)] \succ [L_2 : a, L_3 : (1-a)]$.
Continuity	For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0,1]$, if $L_1 \succ L_2 \succ L_3$, then there exists $a \in (0,1)$ such that $[L_1 : a, L_3 : (1-a)] \sim L_2$

$u : \mathcal{L} \to \Re$ is linear provided for all $L = [L_1 : p_1, \ldots, L_n : p_n] \in \mathcal{L}$,

$$u(L) = \sum_{i=1}^{n} p_i u(L_i)$$

von Neumann-Morgenstern Representation Theorem A binary relation $\succeq$ on $\mathcal{L}$ satisfies Preference, Compound Lotteries, Independence and Continuity iff $\succeq$ is representable by a linear utility function $u : \mathcal{L} \to \Re$.

Moreover, $u' : \mathcal{L} \to \Re$ represents $\succeq$ iff there exists real numbers $c > 0$ and d such that $u'(\cdot) = cu(\cdot) + d$. ("$u$ is unique up to linear transformations.")

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Von Neumann-Morgenstern Theorem. If an agent satisfies the previous axioms, then the agent's ordinal utility function can be turned into cardinal utility function.

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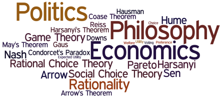
- Utility is unique only *up to linear transformations*. So, it still does not make sense to add two different agents cardinal utility functions.
- Issue with continuity: 1EUR $\succ$ 1 cent $\succ$ death, but who would accept a lottery which is p for 1EUR and $(1 - p)$ for death??
- Important issues about how to identify correct descriptions of the outcomes and options.

Social Utility

Suppose that N is a set of agents and for $i \in N$, u_i is i's cardinal utility function.

Social Utility



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Measures of Social Utility:

- Sum Utilitarian: maximize $\sum_i u_i$
- Average Utilitarian: maximize $\frac{\sum_i u_i}{|N|}$

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Measures of Social Utility:

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- Average Utilitarian: maximize $\frac{\sum_i u_i}{|N|}$
- Egalitarian: maximize $\min_i\{u_i\}$
- Nash: maximize $\Pi_i u_i$

Harsanyi's Theorem

Assume that there is a finite number of citizens ($N = \{1, \ldots, n\}$), and a finite set of social states X.

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Assume that there is a *Planner*.

- The planner's utility function matches the social utility function
- If the Planner is a citizen, he is required to have two (but not necessarily different) preference orderings — his personal ordering and his moral ordering.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

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- Each citizen's preference is represented by a linear utility function u_i
- The Planner's preference is represented by a linear utility function u
- Assume that all the citizens use 0 to 1 utility scales.
- Assume that 0 is the lowest utility scale for the Planner.

Strong Pareto

(P1) For each L, L' if $L \sim_i L'$ for all $i \in N$, then $L \sim L'$

(P2) For each L, L' if $L \succeq_i L'$ for all $i \in N$ and $L \succ_j L'$ for some $j \in N$, then $L \succ L'$

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \Re^n$. That is, the sequence of utility values of L for each agent.

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \mathfrak{R}^n$. That is, the sequence of utility values of L for each agent.

Defined the following two sets:

$$\mathcal{R}^n = \{(r_1, \ldots, r_n) \in \mathfrak{R}^n \mid \text{ there is a } L \in \mathcal{L} \text{ such that for all } i = 1, \ldots, n, u_i(L) = r_i\}$$

and

$$\mathcal{R} = \{r \in \mathfrak{R} \mid \text{there is a } L \in \mathcal{L} \text{ such that } u(L) = r\}$$

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and

$$\mathcal{R} = \{r \in \Re \mid \text{there is a } L \in \mathcal{L} \text{ such that } u(L) = r\}$$

Define a function $f : \mathcal{R}^n \to \mathcal{R}$ as follows: for all $(r_1, \ldots, r_n)$, let $f(r_1, \ldots, r_n) = r$ where $r = u(L)$ with L a lottery such that $(u_1(L), \ldots, u_n(L)) = (r_1, \ldots, r_n)$.

Equity

(E) All agents should be treated equally by the Planner. Formally, this means that $f(r_1, \ldots, r_n) = f(r'_1, \ldots, r'_n)$ when there is a permutation $\pi : N \to N$ such that for each $i = 1, \ldots, n$, $r'_i = r_{\pi(i)}$.

Harsanyi's Theorem For all $(r_1, \ldots, r_n) \in \mathcal{R}^n$, $f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$.

Observation. The function f is well-defined.

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Proof. Suppose that $L, L' \in \mathcal{L}$ such that $(u_1(L), \ldots, u_n(L)) = (u_1(L'), \ldots, u_n(L'))$. Then, for all $i \in N$, i is indifferent between L and L' (i.e., $L \sim_i L'$). Then, by axiom $P1$, we have $L \sim L'$. Thus, $u(L) = u(L')$; and so, f is well-defined.

For each $i \in N$ and $L \in \mathcal{L}$, we have $0 \leq u_i(L) \leq 1$.

For each $i \in N$, let $e_i = (0, 0, \dots, 1, \dots, 0)$ (where there is a 1 in the ith position and 0 everywhere else).

This corresponds to a situation in which a single agent gets her most preferred outcome while all the other agents get their least-preferred outcome.

Lemma. For each $i, j \in N$, $f(e_i) = f(e_j)$

Lemma. For all $a \in \mathfrak{R}$, $af(r_1, \ldots, r_n) = f(ar_1, \ldots, ar_n)$.

Let L be the lottery such that for each $i \in N$, $u_i(L) = r_i$. Consider the lottery $L' = [L : a, \mathbf{0} : (1 - a)]$, where $\mathbf{0}$ is the lottery in which everyone gets their lowest-ranked outcome.

Then, for each $i \in N$, $u_i(\mathbf{0}) = 0$. Furthermore, by the Pareto principle $P1$, we must have $u(\mathbf{0}) = 0$.

Then, for all $i \in N$, we have

1. $u_i(L') = au_i(L) + (1 - a)u_i(\mathbf{0}) = au_i(L) = ar_i$; and
2. $u(L') = au(L) + (1 - a)u(\mathbf{0}) = au(L)$

$$af(r_1, \ldots, r_n) \quad = \quad au(L) \qquad\qquad \text{(definition of } f\text{)}$$

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& = & u(L') & \text{(item 2.)}
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af(r_1, \ldots, r_n) & = & au(L) & \text{(definition of } f\text{)} \\
& = & u(L') & \text{(item 2.)} \\
& = & f(u_1(L'), \ldots, u_n(L')) & \text{(definition of } f\text{)} \\
& = & f(ar_1, \ldots ar_n) & \text{(item 1.)}
\end{array}
$$

Theorem. For all $(r_1, \ldots, r_n) \in \mathcal{R}^n, f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$.

Consider a lottery L such that for all $i \in N$, $u_i(L) = r_i$. Consider lotteries L_i such that $u_i(L_i) = r_i$ and for all $j \neq i$, $u_j(L_i) = 0$. Consider the lottery $L' = [L_1 : 1/n, \ldots, L_n : 1/n]$.

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- $u_i(L') = \sum_{k=1}^{n} \frac{1}{n} u_i(L_k) = \frac{1}{n} u_i(L_i) = \frac{1}{n} r_i$.
- $f(0, \ldots, r_k, \ldots, 0) = r_k f(0, \ldots, 1, \ldots, 0) = r_k$

$$u(L') \quad = \quad \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k)$$

$$
\begin{aligned}
u(L') &= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, f(u_1(L_k), \ldots, u_k(L_k), \ldots, u_n(L_k))
\end{aligned}
$$

$$
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&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, r_k\, f(0, \ldots, 1, \ldots, 0) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, r_k
\end{aligned}
$$

$$u(L') = f(u_1(L'), \ldots, u_n(L'))$$

$$
\begin{aligned}
u(L') &= f(u_1(L'), \ldots, u_n(L')) \\
&= f(\tfrac{1}{n}\, r_1, \ldots, \tfrac{1}{n} r_n)
\end{aligned}
$$

$$
\begin{aligned}
u(L') &= f(u_1(L'), \ldots, u_n(L')) \\
&= f(\tfrac{1}{n}\, r_1, \ldots, \tfrac{1}{n} r_n) \\
&= \tfrac{1}{n} f(r_1, \ldots, r_n)
\end{aligned}
$$

Thus,

$$\frac{1}{n} f(r_1, \ldots, r_k) = u(L') = \sum_{k=1}^{n} \frac{1}{n}\, r_k = \frac{1}{n} \; \sum_{k=1}^{n} r_k$$

Hence, $f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$, as desired.