PHIL309P
Philosophy, Politics and Economics

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- **Final Exam: Thu, May 12 8:00AM - 10:00AM**, EGR 2116 (see Testudo)
    - In-class exam
    - Consult problem sets (Problem sets 2 & 3 will be graded by Thursday or Friday), quizzes
    - Review sheet will be provided on Thursday or Friday
    - Multiple choice, short answers, short essay (questions will be provided).
- Final class: Tuesday, May 10
- Final comment: General reflections about the course, topics you found most interesting, topics you wish we spent more time on, etc.
- A couple quizzes coming.
- I'll be in my office on Wednesday of finals week in case you have questions about the final (you can schedule an appointment to be sure that I'm there).

Fair Division

Suppose that there is a set G of objects that must be divided among a group of individuals.

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Questions:

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    - A set of indivisible objects
    - Several divisible objects
    - A single heterogeneous object

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Suppose that there is a set G of objects that must be divided among a group of individuals.

Questions:

- Are the items *divisible* or *indivisible*?
  - A set of indivisible objects
  - Several divisible objects
  - A single heterogeneous object
- Are side-payments allowed?

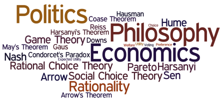
Fair Division

Suppose that there is a set G of objects that must be divided among a group of individuals.

Questions:

- Are the items *divisible* or *indivisible*?
  - A set of indivisible objects
  - Several divisible objects
  - A single heterogeneous object
- Are side-payments allowed?
- Dividing "goods" or "bads"? or both?

- Individual utilities of the goods: Ordinal? Cardinal?



- Individual utilities of the goods: Ordinal? Cardinal?

- Maximize social welfare:
  - Utilitarian: maximize $\sum_i u_i$
  - Egalitarian: maximize $\min_i\{u_i\}$
  - Nash: maximize $\Pi_i u_i$

- Individual utilities of the goods: Ordinal? Cardinal?

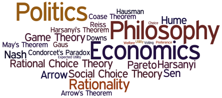
- Maximize social welfare:
  - Utilitarian: maximize $\sum_i u_i$
  - Egalitarian: maximize $\min_i\{u_i\}$
  - Nash: maximize $\Pi_i u_i$

- Preferences over bundles, or allocations: Separable? Additive? Lifted from an ordering over the objects?

Literature

H. Moulin. *Fair Division and Collective Welfare*. The MIT Press, 2003.

S. Brams and A. Taylor. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press, 1998.



Suppose that G is a set of goods to be distributed among n individuals.

An **allocation** is a function $A : N \to \wp(G)$ assigning goods to individuals (note that, in general, it need not be the case that $\bigcup_{i \in N} A(i) = G$).

For each $i \in N$, u_i is i's utility function on *bundles* of goods. Then, the utility of an allocation is $u_i(A) = u_i(A(i))$.

A **profile** of utilities for an allocation A is a tuple $(u_1(A(1)), \dots, u_n(A(n)))$, where $N = \{1, \dots, n\}$ is the set of individuals.

Pareto Efficiency

Suppose that A and A' are allocations.

A **Pareto dominates** A' provided for all $i \in N$, $u_i(A(i)) \geq u_i(A'(i))$ and there is a $j \in N$ such that $u_j(A(j)) > u_j(A'(j))$.

A is **Pareto efficient** if it is not Pareto dominated. (That is, there is no A' such that A' Pareto dominates A)

Envy-Freeness

An allocation A is **envy-free** provided there is no individual i such that

$$u_i(A(j)) > u_i(A(i))$$

for some j.

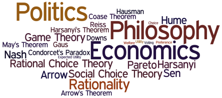
Proportionality

Suppose that G is the set of all the objects and there are n individuals. An allocation A is **proportional** provided for all i:

$$u_i(A) \geq \frac{1}{n} u_i(G)$$

Note that this only makes sense when the utilities are monotonic: for all sets of goods $C \subseteq D \subseteq G$, $u_i(C) \leq u_i(D)$.

Equitability



An allocation A is **equitable** provided for all i, j:

$$u_i(A(i)) = u_j(A(j)))$$

Paradoxes of Fair Division: Indivisible Goods

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, 98:6, pgs. 300-314, 2001.

No Envy-Free Division

$$\begin{array}{lccccc}
\text{Ann:} & 1 & \succ & 2 & \succ & 3 \\
\text{Bob:} & 1 & \succ & 3 & \succ & 2 \\
\text{Cath:} & 2 & \succ & 1 & \succ & 3
\end{array}$$

No Envy-Free Division

$$\begin{array}{lccccc}
\text{Ann:} & 1 & \succ & 2 & \succ & 3 \\
\text{Bob:} & 1 & \succ & 3 & \succ & 2 \\
\text{Cath:} & 2 & \succ & 1 & \succ & 3
\end{array}$$

There are no envy-free divisions.

Envy-Freeness and Efficiency

$$
\begin{array}{rccccccccccc}
\text{Ann:} & 1 & \succ & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
\text{Bob:} & 4 & \succ & 3 & \succ & 2 & \succ & 1 & \succ & 5 & \succ & 6 \\
\text{Cath:} & 5 & \succ & 1 & \succ & 2 & \succ & 6 & \succ & 3 & \succ & 4
\end{array}
$$

Envy-Freeness and Efficiency

Ann:	$\boxed{1}$	$\succ$	2	$\succ$	$\boxed{3}$	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	$\boxed{4}$	$\succ$	3	$\succ$	$\boxed{2}$	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	$\boxed{5}$	$\succ$	1	$\succ$	2	$\succ$	$\boxed{6}$	$\succ$	3	$\succ$	4

Ann: $\{1, 3\}$
Bob: $\{2, 4\}$
Cath: $\{5, 6\}$

Envy-Freeness and Efficiency

Ann:	$\boxed{1}$	$\succ$	$\boxed{2}$	$\succ$	$\boxed{3}$	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	$\boxed{4}$	$\succ$	$\boxed{3}$	$\succ$	$\boxed{2}$	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	$\boxed{5}$	$\succ$	1	$\succ$	2	$\succ$	$\boxed{6}$	$\succ$	3	$\succ$	4

Ann: $\{1, 3\}$
Bob: $\{2, 4\}$
Cath: $\{5, 6\}$

Ann: $\{1, 2\}$
Bob: $\{3, 4\}$
Cath: $\{5, 6\}$

Envy-Freeness and Efficiency

$$\text{Ann:}\quad \boxed{1} \succ \boxed{2} \succ 3 \succ 4 \succ 5 \succ 6$$

$$\text{Bob:}\quad 4 \succ 3 \succ 2 \succ 1 \succ 5 \succ 6$$

$$\text{Cath:}\quad \boxed{5} \succ \boxed{1} \succ \boxed{2} \succ \boxed{6} \succ 3 \succ 4$$

Ann: $\{1,3\}$	Ann: $\{1,2\}$
Bob: $\{2,4\}$	Bob: $\{3,4\}$
Cath: $\{5,6\}$	Cath: $\{5,6\}$

Envy-Freeness and Efficiency

$$\text{Ann: } \boxed{1} \succ \boxed{2} \succ 3 \succ 4 \succ 5 \succ 6$$

$$\text{Bob: } 4 \succ 3 \succ 2 \succ 1 \succ 5 \succ 6$$

$$\text{Cath: } \boxed{5} \succ \boxed{1} \succ \boxed{2} \succ \boxed{6} \succ 3 \succ 4$$

Ann: $\{1, 3\}$	Ann: $\{1, 2\}$
Bob: $\{2, 4\}$	Bob: $\{3, 4\}$
Cath: $\{5, 6\}$	Cath: $\{5, 6\}$

There is no other division that guarantees *envy freeness*

Avoid envy or help the worse off?

$$
\begin{array}{lccccccccccc}
\text{Ann:} & 1 & \succ & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
\text{Bob:} & 5 & \succ & 6 & \succ & 2 & \succ & 1 & \succ & 4 & \succ & 3 \\
\text{Cath:} & 3 & \succ & 6 & \succ & 5 & \succ & 4 & \succ & 1 & \succ & 2
\end{array}
$$

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

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Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
- The only envy-free and efficient division is $(14, 25, 36)$

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Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

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\text{Bob:} & \boxed{5} & \succ & 6 & \succ & \boxed{2} & \succ & 1 & \succ & 4 & \succ & 3 \\
\text{Cath:} & \boxed{3} & \succ & \boxed{6} & \succ & 5 & \succ & 4 & \succ & 1 & \succ & 2
\end{array}
$$

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
- The only envy-free and efficient division is $(14, 25, 36)$

Voting

Ann:	1	≻	2	≻	3	≻	4	≻	5	≻	6
Bob:	5	≻	6	≻	2	≻	1	≻	4	≻	3
Cath:	3	≻	6	≻	5	≻	4	≻	1	≻	2

Allocations	Preferences
A_1: $(12, 56, 34)$	Ann: $A_1 \; I_A \; A_2 \; P_A \; A_3$
A_2: $(12, 45, 36)$	Bob: $A_1 \; P_B \; A_3 \; P_B \; A_2$
A_3: $(14, 25, 36)$	Cath: $A_2 \; I_C \; A_3 \; P_C \; A_1$

Voting

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Preferences
A_1: $(12, 56, 34)$	Ann: $A_1 \; I_A \; A_2 \; P_A \; A_3$
A_2: $(12, 45, 36)$	Bob: $A_1 \; P_B \; A_3 \; P_B \; A_2$
A_3: $(14, 25, 36)$	Cath: $A_2 \; I_C \; A_3 \; P_C \; A_1$

Conclusion: The unique envy-free division would lose in a vote to any of the other efficient divisions

Maximize Total Utility

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Total Utility
A_1: $(12, 56, 34)$	31
A_2: $(12, 45, 36)$	30
A_3: $(14, 25, 36)$	30

Maximize Total Utility

Utility	6		5		4		3		2		1
Ann:	**1**	$\succ$	2	$\succ$	3	$\succ$	**4**	$\succ$	5	$\succ$	6
Bob:	**5**	$\succ$	6	$\succ$	**2**	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	**3**	$\succ$	**6**	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Total Utility
$A_1: (12, 56, 34)$	31
$A_2: (12, 45, 36)$	30
$A_3: (14, 25, 36)$	30

Conclusion: Maximizing the total utility (i.e., the modified Borda score) will not select the unique envy-free division.

Improve the Worse Off

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Minimum Utilities
A_1: $(12, 56, 34)$	$(5, 5, 3)$
A_2: $(12, 45, 36)$	$(5, 2, 5)$
A_3: $(14, 25, 36)$	$(3, 4, 5)$

Improve the Worse Off

Utility	6		5		4		3		2		1
Ann:	$\boxed{1}$	$\succ$	2	$\succ$	3	$\succ$	$\boxed{4}$	$\succ$	5	$\succ$	6
Bob:	$\boxed{5}$	$\succ$	6	$\succ$	$\boxed{2}$	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	$\boxed{3}$	$\succ$	$\boxed{6}$	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Minimum Utilities
A_1: $(12, 56, 34)$	$(5, 5, 3)$
A_2: $(12, 45, 36)$	$(5, 2, 5)$
A_3: $(14, 25, 36)$	$(3, 4, 5)$

Conclusion: (Lexicographic) Maximin will not select the unique envy-free division.

Fair Division

www.spliddit.org

Adjusted Winner

Adjusted Winner

Adjusted winner (*AW*) is an algorithm for dividing n divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor, 1998
- *The Win-Win Solution* by Brams and Taylor, 2000
- `www.nyu.edu/projects/adjustedwinner`
- Fair Outcomes, Inc.: `www.fairoutcomes.com`

Item	Ann	Bob
A		
B		
C		

Suppose Ann and Bob are dividing three goods {A, B, C}

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Suppose Ann and Bob are dividing three goods {A, B, C}

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

Suppose Ann and Bob are dividing three goods {A, B, C}

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

Suppose Ann and Bob are dividing three goods {A, B, C}

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

Equitability Adjustment: Transfer all or part of the goods from the person with the most points until both receive the same number of points

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

Suppose Ann and Bob are dividing three goods $\{A, B, C\}$

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

Equitability Adjustment: Transfer all or part of the goods from the person with the most points until both receive the same number of points

Find the item whose ratio is closes to 1: $65/46 \geq 5/4 \geq 1 \geq 30/50$

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	0	4
B	65	0
C	0	50
Total	65	54

Suppose Ann and Bob are dividing three goods $\{A, B, C\}$

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

Equitability Adjustment: Transfer all or part of the goods from the person with the most points until both receive the same number of points

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Item	Ann	Bob
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Total	65	54

Suppose Ann and Bob are dividing three goods {A, B, C}

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

Equitability Adjustment: Transfer all or part of the goods from the person with the most points until both receive the same number of points

Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$ yielding $p = \frac{100}{111} = 0.901$

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	0	4
B	58.56	4.56
C	0	50
Total	58.56	58.56

Suppose Ann and Bob are dividing three goods $\{A, B, C\}$

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

Equitability Adjustment: Transfer all or part of the goods from the person with the most points until both receive the same number of points

Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$ yielding $p = \frac{100}{111} = 0.901$

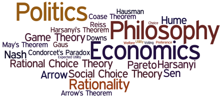
Easy Observations

- For two-party disputes, proportionality and envy-freeness are equivalent.
- *AW* only produces equitable allocations (equitability is essentially built in to the procedure).
- *AW* produces allocations in which at most one good is split.

Adjusted Winner is Fair

Theorem (Brams and Taylor) *AW produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations).*

Strategizing



In Adjusted Winner, can the people improve their allocation by misrepresenting their preferences?

Strategizing

In Adjusted Winner, can the people improve their allocation by misrepresenting their preferences?

Yes

Strategizing: Example

Item	Ann	Bob
Matisse	75	25
Picasso	25	75

Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.

Strategizing: Example

Suppose Ann knows Bob’s preferences, but Bob does not know Ann’s.

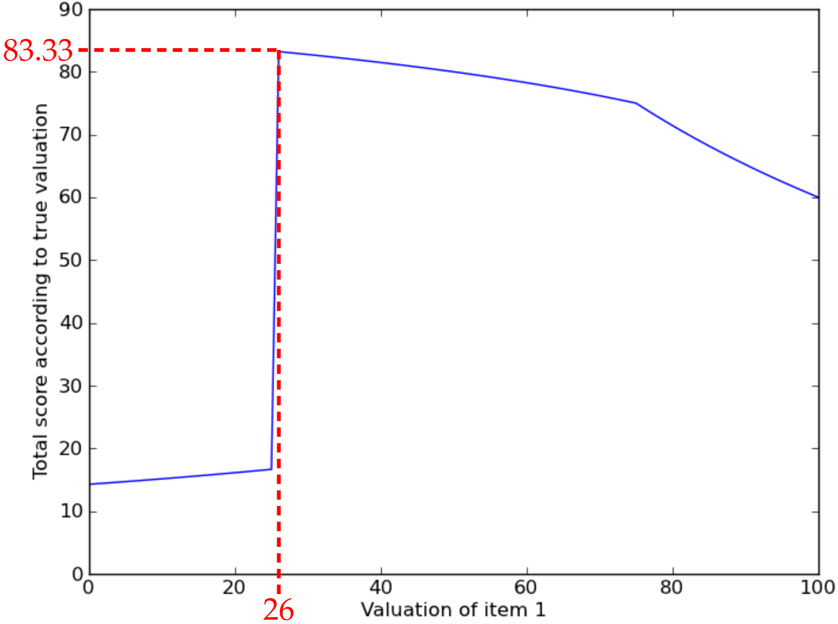
Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	25
P	74	75

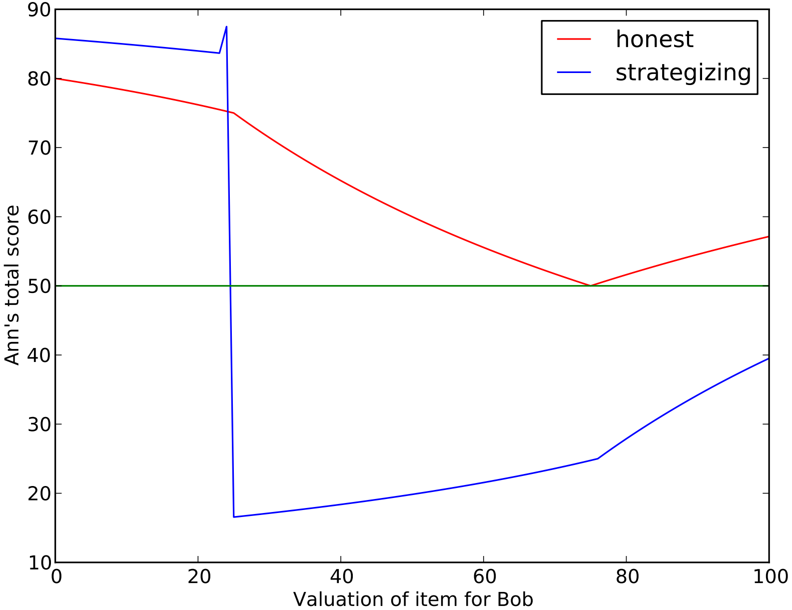
So Ann will get M plus a portion of P.

According to Ann’s announced allocation, she receives 50.33 points

According to Ann’s actual allocation, she receives $75 + 0.33 * 25 = 83.33$ points.


83.33
26
Total score according to true valuation
Valuation of item 1
0
10
20
30
40
50
60
70
80
90
0
20
40
60
80
100

However, while honesty may not always be the best policy it is the only **safe** one, i.e., the only one which will guarantee 50%.

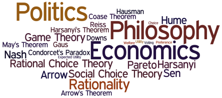


honest
strategizing
Ann's total score
Valuation of item for Bob
90
80
70
60
50
40
30
20
10
0
20
40
60
80
100

Main Question

How do we cut a cake fairly?

Main Question



How do we cut a cake fairly?

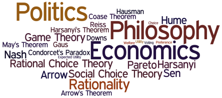
- A cake is a metaphor for a divisible heterogeneous good.

Main Question

How do we cut a cake fairly?

- We are interested not only in the *existence* of a (fair) division but also a *constructive procedure* (an algorithm) for finding it
  - discrete procedures
  - continuous moving knife procedures

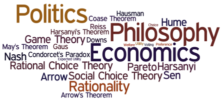
Main Question



How do we cut a cake fairly?

- Different results known for 2,3,4,... cutters!

Main Question



How do we cut a cake fairly?

- Many ways to make this precise!

Main References

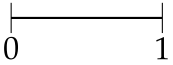
S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. 1996.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair If You Can*. 1998.

J. Barbanel. *The Geometry of Efficient Fair Division*. 2005.

The Cake-Cutting Problem

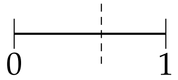
The cake is the unit interval $[0, 1]$



The Cake-Cutting Problem

The cake is the unit interval $[0, 1]$

Only parallel, vertical cuts, perpendicular to the horizontal x-axis are made



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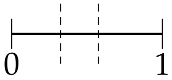
Only parallel, vertical cuts, perpendicular to the horizontal x-axis are made



The Cake-Cutting Problem

Each player i has a continuous value measure $v_i(x)$ on $[0, 1]$ such that

- $v_i(x) \geq 0$ for $x \in [0, 1]$
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value of finite number of disjoint pieces equals the value of their union (hence, no subpieces have greater value than the larger piece containing them).

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a single cut (which defines the border of a piece) has no area and so has no value.

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no portion of cake is of positive measure for one player and zero measure for another player.

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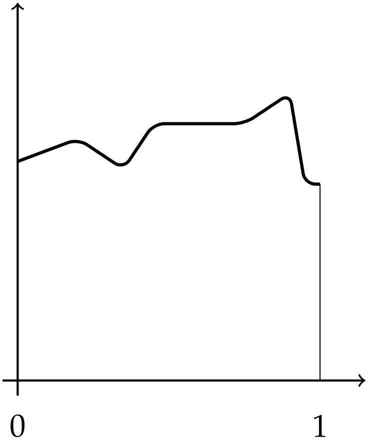

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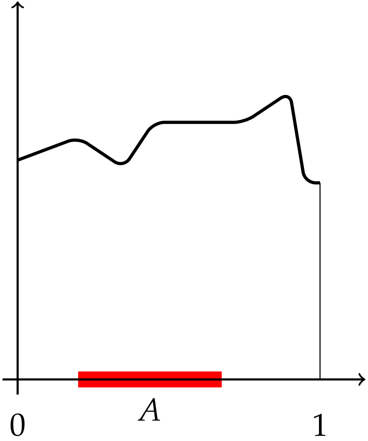

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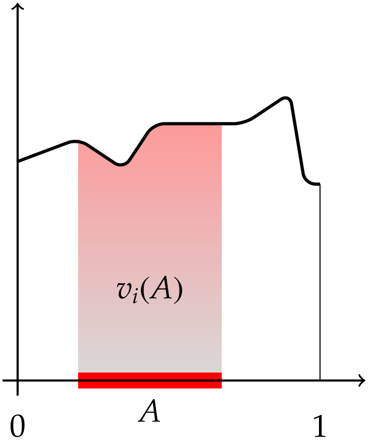

Fairness

A division of a cake $[0,1]$ for n players is a partition $(S_1, \ldots, S_n)$ (i.e., each $S_i \subseteq [0,1]$, $\cup_i S_i = [0,1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0,1]$).

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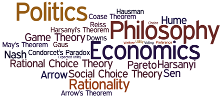

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- **Envy-Free**: for each i,j, $v_i(S_i)\geq v_i(S_j)$
- **Equitable**: for each i,j, $v_i(S_i)=v_j(S_j)$
- **Efficient**: there is no other division $(T_1,\ldots,T_n)$ such that $v_i(T_i)\geq v_i(S_i)$ for all i and there is some j such that $v_j(T_j)>v_j(S_j)$.

Truthfulness

Some procedures ask players to represent their preferences.

This representation need not be "truthful"

Typically, it is assumed that agents will follow a maximin strategy (maximize the set of items that are guaranteed)

Two Players

Procedure: one player cuts the cake into two portions and the other player chooses one of the portions.

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This strategy creates an **envy-free** and **efficient** allocation, but it is not necessarily **equitable**.

Example

Suppose that the cake is half chocolate and have vanilla.
Ann values the vanilla half twice as much as the chocolate half:

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

Bob values both sides equally:

$$v_B(x) = \begin{cases} 1 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$$

Where should A cut the cake?

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A should cut the cake at $x = 3/8$:

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Note that the portions are not equitable (B receive 5/8 according to his valuation)

Cut and Choose is not Equitable

Suppose A values the vanilla half twice as much as the chocolate half:

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The Surplus Procedure

S. Brams, M. A. Jones and C. Klamler. *Better Ways to Cut a Cake*. Notices of the AMS, 53:11, pgs. 1314-1321, 2006.

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4. Suppose a is to the left of b (Then A receives $[0, a]$ and B receives $[b, 1]$). Cut the cake a point c in $[a, b]$ at which the players receive the *same proportion* p of the cake in this interval.

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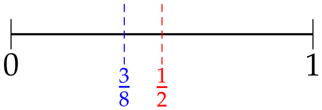

Which Cut-Point?

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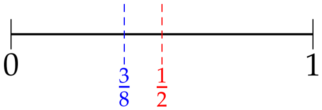


Proportional equitability: $c = \frac{7}{16}$

Equitability: $e = \frac{3}{7}$

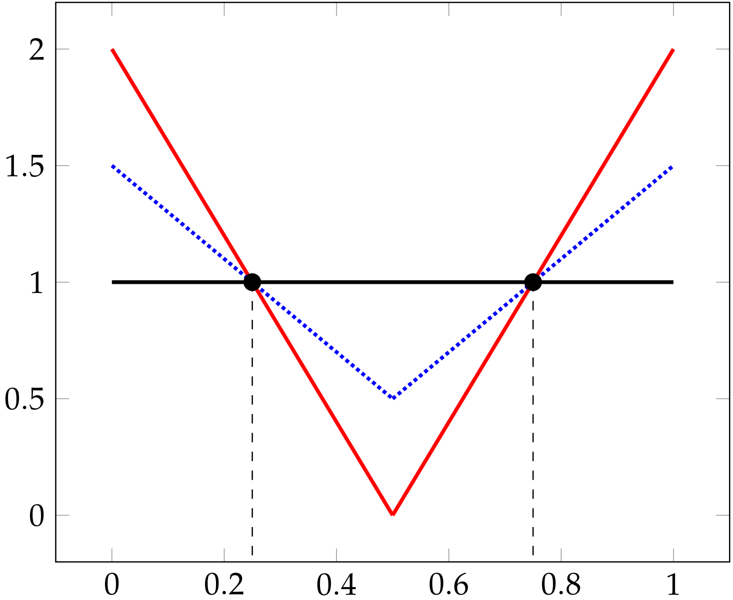
Surplus Procedure

A procedure is **strategy-proof** if maximin players always have an incentive to let $f_A = v_A$ and $f_B = v_B$.

Theorem. The Surplus Procedure (with the proportional equitability cut-point c) is strategy-proof, whereas any procedure that makes e the cut-point is strategy-vulnerable.

More than 2 Players

Fact. It is not always possible to divide a cake among three players into **envy-free and equitable** portions using 2 cuts.


2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1