Some Notes on Propositional and First Order Logic

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1 Propositional Logic

Suppose that At is a (finite or countable) set of **atomic propositions**. Propositional formulas are defined inductively:

- If $p \in \mathsf{At}$, then p is a propositional formula.
- If φ is a propositional formulas, then so is $\neg\varphi$.
- If φ, ψ are propositional formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$.
- Nothing else is a propositional formula.

Rather than writing out the full inductive definition, it is common to define a formal language by specifying the (context-free) grammar that generates the language:

Definition 1 (Propositional Formulas) Suppose that At is a set of atomic propositions. Let $\mathcal{L}(\mathsf{At})$ be the smallest set of formulas defined by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi$$

where $p \in \mathsf{At}$. We write $\mathcal{L}$ instead of $\mathcal{L}(\mathsf{At})$ when the set of atomic propositions is understood. ◁

Definition 2 (Propositional Valuation) A propositional valuation is a function $V : \mathsf{At} \rightarrow \{1, 0\}$. This function is extended to all propositional formulas, denoted $\overline{V} : \mathcal{L}(\mathsf{At}) \rightarrow \{0, 1\}$, as follows:

- $\overline{V}(p) = V(p)$ for all $p \in \mathsf{At}$

- $\overline{V}(\neg\varphi) = \begin{cases} 1 & \text{if } \overline{V}(\varphi) = 0 \\ 0 & \text{if } \overline{V}(\varphi) = 1 \end{cases}$

- $\overline{V}(\varphi \wedge \psi) = \begin{cases} 1 & \text{if } \overline{V}(\varphi) = 1 \text{ and } \overline{V}(\psi) = 1 \\ 0 & \text{otherwise} \end{cases}$

- $\overline{V}(\varphi \vee \psi) = \begin{cases} 1 & \text{if } \overline{V}(\varphi) = 1 \text{ or } \overline{V}(\psi) = 1 \\ 0 & \text{otherwise} \end{cases}$

- $\overline{V}(\varphi \to \psi) = \begin{cases} 0 & \text{if } \overline{V}(\varphi) = 1 \text{ and } \overline{V}(\psi) = 0 \\ 1 & \text{otherwise} \end{cases}$

To simplify the notation, we often write V for both the propositional valuation and its extension to the full set of propositional formulas. ◁

Sometimes it is convenient to include two special atomic propositions '$\bot$' and '$\top$', meaning 'false' and 'true', respectively. We can either think of these atomic proposition as being defined ($\bot$ is $p \wedge \neg p$ and $\top$ is $p \vee \neg p$ for some $p \in \mathsf{At}$) or as special atomic propositions where for all propositional valuations, $V(\bot) = 0$ and $V(\top) = 1$.

We say that a set Γ of propositional formulas is **satisfiable** if all the formulas in Γ can be true at the same time, i.e., there is a propositional valuation V such that for all $\varphi \in \Gamma$, $V(\varphi) = 1$. A formula $\varphi \in \Gamma$ is **valid** if for all propositional valuations V, $V(\varphi) = 1$.

Definition 3 (Logical Consequence) Suppose that Γ is a set propositional formulas. We say that φ is a logical consequence of Γ, denoted $\Gamma \models \varphi$, provided that for all propositional valuations V, if for all $\psi \in \Gamma$, $V(\psi) = 1$, then $V(\varphi) = 1$. ◁

There are many different types of axiomatizations for propositional logic (e.g., Hilbert-style deductions, natural deduction systems, Gentzen systems, Tableaux). Consider the following *axiom schemes* and rule:

1. $\alpha \to (\beta \to \alpha)$
2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
3. $\bot \to \alpha$
4. $(\alpha \wedge \beta) \to \alpha$
5. $(\alpha \wedge \beta) \to \beta$
6. $\alpha \to (\beta \to (\alpha \wedge \beta))$
7. $\alpha \to (\alpha \vee \beta)$
8. $\beta \to (\alpha \vee \beta)$
9. $(\alpha \to \bot) \to ((\beta \to \bot) \to ((\alpha \vee \beta) \to \bot)$
10. $((\alpha \to \bot) \to \bot) \to \alpha$
11. (Modus Ponens) $\dfrac{\alpha \quad \alpha \to \beta}{\beta}$

Note that α, β and γ should be thought of as meta-variables that will be replaced with any formula of propositional logic.

Definition 4 (Deduction) Suppose that Γ is a set of propositional formulas. A deduction of φ from Γ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ where $\varphi_n = \varphi$, for each $i = 1, \ldots, n$, φ_i is either an element of Γ, an instance of one of the above axiom schemes or follows from earlier formulas by Modus Ponens (i.e., there are φ_j, φ_k such that $j, k < i$, $\varphi_j = \alpha$, $\varphi_k = \alpha \to \beta$ and $\varphi_i = \beta$. We write $\Gamma \vdash \varphi$ when there is a deduction of φ from Γ. ◁

We say that a set of formulas Γ is **consistent** if $\Gamma \nvdash \bot$. The two key theorems relating deductions and logical consequence are Soundness and Completeness:

Theorem 5 (Soundness) *$\Gamma \vdash \varphi$ implies that $\Gamma \models \varphi$.*

Theorem 6 (Completeness) *$\Gamma \models \varphi$ implies that $\Gamma \vdash \varphi$.*

1.1 Possible Worlds

Suppose that W is a non-empty set, elements of which are called **possible worlds**, or **states**. Each possible world is associated with a propositional valuation. This is typically expressed by a **valuation function**: $V : W \times \mathsf{At} \to \{0, 1\}$. A valuation function is extended to a function $\overline{V} : W \times \mathcal{L} \to \{0, 1\}$ as in Definition 2. As above, we ofter write $V : W \times \mathcal{L} \to \{0, 1\}$ for both the valuation function and its extension to $\mathcal{L}$.

Each valuation function $V : W \times \mathcal{L} \to \{0, 1\}$ is associate with a function $[\![\cdot]\!] : \mathcal{L} \to \wp(W)$, where $\wp(W)$ is the set of all subsets of W, as follows:

$$\text{For each } \varphi \in \mathcal{L},\ [\![\varphi]\!] = \{w \mid V(w, \varphi) = 1\}$$

It is a straightforward (but instructive!) exercise to verify the following Fact:

Fact 7 *For all $\varphi \in \mathcal{L}$,*

- $[\![\neg\varphi]\!] = W - [\![\varphi]\!]$
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$
- $[\![\varphi \to \psi]\!] = (W - [\![\varphi]\!]) \cup [\![\psi]\!]$

2 First-Order Logic

The language of predicate logic is constructed from a number of different pieces of syntax: variables, constants, function symbols and predicate symbols. Both function and predicate symbols are associated with an *arity*: the number of arguments that are required by the function or predicate. We start by defining **terms**. Let $\mathcal{V}$ be a finite (or countable) set of **variables** and C a set of **constants**.

Definition 8 (Terms) Let $\mathcal{V}$ be a set of variable, C a set of constant symbols and $\mathcal{F}$ a set of function symbols. Each function symbol is associated with an **arity** (a positive integer specifying the number of arguments). Write $f^{(n)}$ when the arity of f is n. A term τ is constructed as follows:

- Any variable $x \in \mathcal{V}$ is a term.
- Any constant $c \in C$ is a term.
- If $f^{(n)} \in \mathcal{F}$ is a function symbol (i.e., f accepts n arguments) and $\tau_1, \ldots, \tau_n$ are terms, then $f(\tau_1, \ldots, \tau_n)$ is a term.
- Nothing else is a term.

Let $\mathcal{T}$ be the set of terms. $\triangleleft$

Terms are used to construct atomic formulas:

Definition 9 (Atomic Formulas) Let $\mathcal{P}$ be a set of predicate symbols. Each predicate symbol is associated with an arity (the number of objects that are related by P). We write $P^{(n)}$ if the arity of P is n. Suppose that P is an atomic predicate symbol with arity n. If $\tau_1, \ldots, \tau_n$ are terms, then $P(\tau_1, \ldots, \tau_n)$ is an atomic formula. To simplify the notation, we may write $P\tau_1\tau_2 \cdots \tau_n$. A special predicate symbol '=' is included with the intended interpretation *equality*. ◃

Definition 10 (Formulas) Formulas are constructed as follows:

- Atomic formulas $P(\tau_1, \ldots, \tau_n)$ are formulas;
- If φ is a formula, then so is $\neg\varphi$;
- If φ and ψ are a formulas, then so is $\varphi \wedge \psi$;
- If φ is a formula, then so is $(\forall x)\varphi$, where x is a variable;
- Nothing else is a formula.

The other boolean connectives ($\vee, \rightarrow, \leftrightarrow$) are defined as usual. In addition, $(\exists x)\varphi$ is defined as $\neg(\forall x)\neg\varphi$. ◃

Definition 11 (Free Variable) Suppose that x is a variable. Then, x **occurs free in** φ is defined as follows:

1. If φ is an atomic formula, then x occurs free in φ provided x occurs in φ (i.e., is a symbol in φ).
2. x occurs free in $\neg\psi$ iff x occurs free in ψ
3. x occurs free in $\psi_1 \wedge \psi_2$ iff x occurs free in ψ_1 or x occurs free in ψ_2
4. x occurs free in $(\forall y)\psi$ iff x occurs free in ψ and $x \neq y$
5. x occurs free in $(\exists y)\psi$ iff x occurs free in ψ and $x \neq y$ ◃

The set of free variables in φ, denoted $\mathsf{Fr}(\varphi)$, is defined by recursion as follows:

1. If φ is an atomic formula, then $\mathsf{Fr}(\varphi)$ is the set of all variables (if any) that occur in φ
2. If φ is $\neg\psi$, then $\mathsf{Fr}(\neg\varphi) = \mathsf{Fr}(\varphi)$
3. If φ is $\psi_1 \wedge \psi_2$, then $\mathsf{Fr}(\varphi) = \mathsf{Fr}(\psi_1) \cup \mathsf{Fr}(\psi_2)$
4. If φ is $(\forall x)\psi$, then $\mathsf{Fr}(\psi) = \mathsf{Fr}(\psi)$ after removing x, if present.

A variable x that is not free is said to be **bound**. Formulas that do not contain any free variables are called sentences:

Definition 12 (Sentence) If φ is a formula and $\mathsf{Fr}(\varphi) = \emptyset$ (i.e., there are no free variables), then φ is a **sentence**. ◃

2.1 Substitutions

If τ and τ' are terms, we write $\tau[x/\tau']$ for the terms where x is replaced by τ'. We can formally define this operation by recursion:

- $x[x/\tau'] = \tau'$
- $y[x/\tau'] = y$ for $x \neq y$
- $c[x/\tau'] = c$
- $F(\tau_1, \ldots, \tau_n)[x/\tau'] = F(\tau_1[x/\tau'], \ldots, \tau_n[x/\tau'])$

The same notation can be used for formulas $\varphi[x/\tau]$ which means replace all free occurrences of x with τ in a formula φ. This is defined as follows:

- $P(\tau_1, \ldots, \tau_n)[x/\tau] = P(\tau_1[x/\tau], \ldots, \tau_n[x/\tau])$
- $\neg\psi[x/\tau] = \neg(\varphi[x/\tau])$
- $(\psi_1 \wedge \psi_2)[x/\tau] = \psi_1[x/\tau] \wedge \psi_2[x/\tau]$
- $((\forall x)\varphi)[x/\tau] = (\forall x)\varphi$
- $((\forall y)\varphi)[x/\tau] = (\forall y)\varphi[x/\tau]$, where $y \neq x$

The following are key examples of this operation:

1. $(x = y)[y/x]$ is $x = x$ and $(x = y)[x/y]$ is $y = y$,
2. $(\forall x(x = y))[x/y]$ is $(\forall x)x = y$,
3. $(\forall x(x = y))[y/x]$ is $(\forall x)x = x$,
4. $(\forall x)\neg(\forall y)(x = y) \rightarrow (\neg\forall y(x = y))[x/y]$ is $(\forall x)\neg(\forall y)(x = y) \rightarrow \neg\forall y(y = y)$.

Definition 13 (Substitutability) A term τ is **substitutable for** x **in** φ is defined as follows:

- For an atomic formula φ, τ is always substitutable for x in φ (there are no quantifiers, so t can always be substituted for x)
- τ is substitutable for x in $\neg\psi$ iff τ is is substitutable for x in ψ
- τ is substitutable for x in $\psi_1 \wedge \psi_1$ iff τ is is substitutable for x in ψ_1 and τ is is substitutable for x in ψ_2
- τ is substitutable for x in $(\forall y)\psi$ iff either
  1. x does not occur free in $(\forall y)\psi$
  2. y does not occur in τ and τ is substitutable for x in ψ. ◁

2.2 First-Order Models

2.2.1 Interpreting Terms

Suppose that W is a set. An **interpretation** I (for W) associates with each functions symbol F a function on W of the appropriate arity, denoted F^I, and to each constant c an element of W, denoted c^I. If W is a set and I an interpretation, then for a function symbol F of arity n,

$$F^I : \underbrace{W \times \cdots \times W}_{n \text{ times}} \to W$$

For each constant symbol, c, we have

$$c^I \in W$$

Our goal is to show how to associate with each term and element of a set W. We first need the notion of a substitution:

Definition 14 (Substitution) Suppose that W is a nonempty set. A **substitution** is a function $\mathbf{s} : \mathcal{V} \to W$. ◃

Definition 15 (Interpretation of Terms) Suppose that I is an interpretation for W and $\mathbf{s} : \mathcal{V} \to W$ is a substitution. We define the function $(I, \mathbf{s}) : \mathcal{T} \to W$ by recursion as follows:

- $(I, \mathbf{s})(x) = \mathbf{s}(x)$
- $(I, \mathbf{s})(c) = c^I$
- $(I, \mathbf{s})(F(\tau_1, \dots, \tau_n)) = F^I((I, \mathbf{s})(\tau_1), \dots, (I, \mathbf{s})(\tau_n))$ ◃

Suppose that $\mathbf{s} : \mathcal{V} \to W$ is a substitution. If $a \in W$, we define a new substitution $\mathbf{s}[x/a]$ as follows:

$$\mathbf{s}[x/a](y) = \begin{cases} a & \text{if } y = x \\ \mathbf{s}(x) & \text{otherwise} \end{cases}$$

Suppose that $\mathbf{s} : \mathcal{V} \to W$ and $\mathbf{s}' : \mathcal{V} \to W$ are two substitutions. For each variable $x \in \mathcal{V}$, we define a relation on the set of substitutions as follows:

$$\mathbf{s} \sim_x \mathbf{s}' \text{ iff } \mathbf{s}(y) = \mathbf{s}'(y) \text{ for all } y \neq x$$

Hence, $\mathbf{s} \sim_x \mathbf{s}'$ provided there is some $a \in W$ such that $\mathbf{s}' = \mathbf{s}[x/a]$.

2.2.2 First Order Models

Definition 16 (Model) A model is a pair $\mathfrak{A} = \langle W, I \rangle$ where W is a nonempty set (called the domain) and I is a function (called the interpretation) assigning to each function symbol F, a function denoted F^I, to each constant symbol, an element of W denoted c^I and to each predicate symbol P, a relation on W of the appropriate arity. If P has arity n, then we have

$$P^I \subseteq \underbrace{W \times \cdots \times W}_{n \text{ times}}$$

If $\mathcal{A}$ is a model, we write $|\mathcal{A}|$ for the domain of $\mathcal{A}$, and we write $F^{\mathcal{A}}$, $c^{\mathcal{A}}$ and $P^{\mathcal{A}}$ to denote F^I, c^I and P^I, respectively. ◁

We say $\mathbf{s}$ is a substitution for $\mathcal{A}$ provided $\mathbf{s} : \mathcal{V} \to |\mathcal{A}|$. Let $\mathcal{A} = \langle W, I \rangle$ be a model. For each term τ, we write $\tau^{\mathcal{A},\mathbf{s}}$ for $(I, \mathbf{s})(\tau)$.

Definition 17 (Truth) Suppose that $\mathcal{A}$ is a model and $\mathbf{s}$ is a substitution for $\mathcal{A}$. The formula φ is true in $\mathcal{A}$ (given $\mathbf{s}$), denoted $\mathcal{A}, \mathbf{s}\varphi$, is defined by recursion as follows:

- $\mathcal{A}, \mathbf{s} \models P(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^{\mathcal{A},\mathbf{s}}, \ldots, \tau_n^{\mathcal{A},\mathbf{s}}) \in P^{\mathcal{A}}$
- $\mathcal{A}, \mathbf{s} \models \neg\psi$ iff $\mathcal{A}, \mathbf{s} \not\models \psi$
- $\mathcal{A}, \mathbf{s} \models \psi_1 \wedge \psi_2$ iff $\mathcal{A}, \mathbf{s} \models \psi_1$ and $\mathcal{A}, \mathbf{s} \models \psi_2$
- $\mathcal{A}, \mathbf{s} \models (\forall x)\psi$ iff for all substitutions $\mathbf{s}'$ for $\mathcal{A}$ if $\mathbf{s} \sim_x \mathbf{s}'$, then $\mathcal{A}, \mathbf{s}' \models \psi$ ◁

2.3 Deductions in First Order Logic

An axiom system for first-order logic consists of the following four axioms (there are others, this is the one from Enderton's *Introduction to Mathematical Logic*):

1. All tautologies
2. $(\forall x)\varphi \to \varphi[x/t]$, where τ is substitutable for x in φ
3. $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
4. $\varphi \to (\forall x)\varphi$, where x does not occur free in φ

Definition 18 (Generalization) Given a formula φ, a **generalization of** φ is a formula of the form $(\forall x_1) \cdots (\forall x_n)\varphi$. ◁

Definition 19 (Tautology) A tautology (in FOL) is any formula obtained by replacing each atomic proposition with a first-order formula. ◁

Definition 20 (Deduction) We write $\Gamma \vdash \varphi$ iff there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$, each φ_i is either a generalization of one of the above axioms, is an element of Γ, or follows from earlier formulas on the list by modus ponens. We write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$. ◁

Example . $\vdash \exists x(\alpha \wedge \beta) \to \exists x\alpha \wedge \exists x\beta$.

1.	$\forall x(\neg\alpha \to \neg(\alpha \wedge \beta))$	Instance of Axiom 1
2.	$\forall x(\neg\alpha \to \neg(\alpha \wedge \beta)) \to (\forall x\neg\alpha \to \forall x\neg(\alpha \wedge \beta))$	Instance of Axiom 3
3.	$\forall x\neg\alpha \to \forall x\neg(\alpha \wedge \beta)$	MP 1,2
4.	$(\forall x\neg\alpha \to \forall x\neg(\alpha \wedge \beta)) \to (\neg\forall x\neg(\alpha \wedge \beta) \to \neg\forall x\neg\alpha)$	Instance of Axiom 1
5.	$\neg\forall x\neg(\alpha \wedge \beta) \to \neg\forall x\neg\alpha$	MP 3,4
6.	$\exists x(\alpha \wedge \beta) \to \exists x\alpha$	Definition of '$\exists$'
7.	$\forall x(\neg\beta \to \neg(\alpha \wedge \beta))$	Instance of Axiom 1
8.	$\forall x(\neg\beta \to \neg(\alpha \wedge \beta)) \to (\forall x\neg\beta \to \forall x\neg(\alpha \wedge \beta))$	Instance of Axiom 3
9.	$\forall x\neg\beta \to \forall x\neg(\alpha \wedge \beta)$	MP 7,8
10.	$(\forall x\neg\beta \to \forall x\neg(\alpha \wedge \beta)) \to (\neg\forall x\neg(\alpha \wedge \beta) \to \neg\forall x\neg\beta)$	Instance of Axiom 1
11.	$\neg\forall x\neg(\alpha \wedge \beta) \to \neg\forall x\neg\beta$	MP 9,10
12.	$\exists x(\alpha \wedge \beta) \to \exists x\beta$	Definition of '$\exists$'
13.	$(\exists x(\alpha \wedge \beta) \to \exists x\alpha) \to ((\exists x(\alpha \wedge \beta) \to \exists x\beta)$ $\to (\exists x(\alpha \wedge \beta) \to (\exists x\alpha \wedge \exists x\beta)))$	Instance of Axiom 1
14.	$(\exists x(\alpha \wedge \beta) \to \exists x\beta) \to (\exists x(\alpha \wedge \beta) \to (\exists x\alpha \wedge \exists x\beta))$	MP 6,13
15.	$\exists x(\alpha \wedge \beta) \to (\exists x\alpha \wedge \exists x\beta)$	MP 12, 14

2.4 Basic Model Theory

- A set of formulas T is **inconsistent** provided $T \vdash \bot$ (where $\bot$ is a formula of the form $\mathbf{0} \neq \mathbf{S}(\mathbf{0})$. A set of formulas T is **consistent** if it is not inconsistent.

- Suppose that T is a set of sentences. Then $Cn(T) = \{\varphi \mid T \vdash \varphi\}$ is the set of (first-order) **consequences** of T.

- Suppose that $\mathcal{A}$ is a first-order model. Then, $Th(\mathcal{A}) = \{\varphi \mid \varphi$ is a sentence and $\mathcal{A} \models \varphi\}$ is the **theory of** $\mathcal{A}$. For example, $Th(\mathcal{N}_S)$ is the set of sentences of $\mathcal{L}_S$ true in $\mathcal{N}_S$; and $Th(\mathcal{N})$ is the set of sentences of $\mathcal{L}_A$ true in $\mathcal{N}$ (the **theory of true arithmetic**).

- A set of sentences T is **satisfiable** if there is a model $\mathcal{A}$ such that $\mathcal{A} \models T$ (where $\mathcal{A} \models T$ means $\mathcal{A} \models \varphi$ for each $\varphi \in T$).

- A **theory** is a set of sentences.

  A **theory** is (effectively) axiomatizable provided there is recursive set A of sentences (and possibly rules) such that $Cn(A) = T$. A theory T is **finitely axiomatizable** provided there is a finite set A of sentences (and possibly rules) such that $Cn(A) = T$.

A theory T (in the language $\mathcal{L}$) is **negation-complete** provided for every sentence of φ in $\mathcal{L}$, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

A theory T is **decidable** provided the set $Cn(T)$ is recursive.

Some useful observations and Theorems:

- If $\mathcal{L}$ is a first-order language constructed from a signature of size κ (where κ is a cardinal), then $|\mathcal{L}| = \max\{\aleph_0, \kappa\}$ ($\aleph_0$ is the first countable cardinal). Thus, there are countably many formulas of $\mathcal{L}_A$.
- The set $\mathcal{L}$ of well-formed formulas (wff) is recursive.
- If T is effectively axiomatizable, then $Cn(T)$ is semidecidable.
- If T is effectively axiomatizable and negation-complete, then $Cn(T)$ is decidable.
- *Model Construction Theorem*. Every consistent set of formulas has a model.
- *Compactness Theorem*. If every finite subset of T is satisfiable, then T is satisfiable.
- *Löwenheim-Skolem Theorem*. If T has a model, then T has a countable model. A model $\mathcal{A}$ is countable provided the domain of $\mathcal{A}$ is countable (i.e., $|\mathcal{A}|$ is countable). The upward Löwenheim-Skolem Theorem states that if T has a model, then it has a model of any infinite cardinality κ.

Two structures $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent**, denoted $\mathcal{A} \equiv \mathcal{B}$, provided for every sentence φ, $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$ (i.e., $Th(\mathcal{A}) = Th(\mathcal{B})$).

Definition 21 (Isomorphism) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two models. A function $f : |\mathcal{A}| \to |\mathcal{B}|$ is an **isomorphism** provided

- f is a bijection
- For all constants $c \in C$, $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$
- $f(F^{\mathcal{A}}(a_1, \ldots, a_n)) = F^{\mathcal{B}}(f(a_1), \ldots, f(a_n))$
- For all $(a_1, \ldots, a_n) \in P^{\mathcal{A}}$ iff $(f(a_1), \ldots, f(a_n) \in P^{\mathcal{B}}$

We write $\mathcal{A} \cong \mathcal{B}$ when there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. ◁

Isomorphism Theorem. For any two first-order models if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.

There are examples of structures that are elementarily equivalent but not isomorphic (e.g., $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ cannot be distinguished by a first-order formula, but are not isomorphic since there is no bijection function from $\mathbb{R}$ to $\mathbb{Q}$.)

Suppose that $\mathcal{A}$ is a first-order structure. A set $X \subseteq |\mathcal{A}|$ is **definable** (in the language $\mathcal{L}$) provided there is a formula $\varphi(x)$ with one free variable such that

$$X = \{a \mid \mathcal{A} \models \varphi(a)\}$$

This definition can be readily adapted to k-ary relations $X \subseteq |\mathcal{A}|^k$.

Example. $\mathbb{N}$ is not definable in the structure $(\mathbb{R}, <)$. Suppose it is defined by $\varphi(x)$ in the first-order language with equality and $<$. Consider $h : \mathbb{R} \to \mathbb{R}$ defined as $h(r) = r^3$. Then, h is a isomorphism between $(\mathbb{R}, <)$ and itself (it is an *automorphism*). Thus, by the Isomorphism Theorem, $(\mathbb{R}, <) \models \varphi(r)$ iff $(\mathbb{R}, <) \models \varphi(h(r))$. But, then $\sqrt[3]{2} \notin \mathbb{N}$ implies $(\mathbb{R}, <) \not\models \varphi(\sqrt[3]{2})$ iff $(\mathbb{R}, <) \not\models \varphi(h(\sqrt[3]{2}))$ iff $(\mathbb{R}, <) \not\models \varphi(2)$, which is a contradiction since $2 \in \mathbb{N}$.