Lecture 3: Frame Definability

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February 8, 2019

1 Bisimulation Review

- **Tree Unfoldings**: The unfolding of $\mathcal{M} = \langle W, R, V\rangle$ with root w is $\vec{\mathcal{M}} = \langle \vec{W}, \vec{R}, \vec{V}\rangle$, where $\vec{W}$ is the set of paths starting at w, $(w, \ldots, w_n) \vec{R} (w, \ldots, w_n, w_{n+1})$ iff $w_n R w_{n+1}$ and $(w, \ldots, w_n) \in V(p)$ iff $w_n \in V(p)$.

  Lemma. Tree-model property: If a formula is satisfiable, then it is satisfiable on a tree structure.

- **Bisimulation**: A bisimulation between $\mathcal{M} = \langle W, R, V\rangle$and $\mathcal{M}' = \langle W', R', V'\rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

  Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$
  Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
  Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv

  – *We write* $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ *if there is a* Z *such that* wZw'.
  – *We write* $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ *iff for all* $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ *iff* $\mathcal{M}', w' \models \varphi$.
  – **Lemma** If $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$.
  – **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$.
  – **Lemma** On m-**saturated models**, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$.

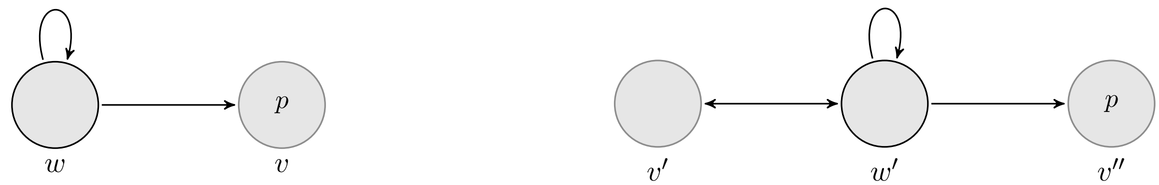
Proposition. Any Kripke structure is the bounded morphic image of a disjoint union of rooted Kripke structures (in fact, tree structures).

Defining classes of models/frames

- $PKS(\varphi) = \{(\mathcal{M}, w) \mid \mathcal{M}, w \models \varphi\}$
- $KS(\varphi) = \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$
- $PFR(\varphi) = \{(\mathcal{F}, w) \mid (\mathcal{F}, V), w \models \varphi$ for all valuations $V\}$
- $FR(\varphi) = \{\mathcal{F} \mid (\mathcal{F}, V), w \models \varphi$ for all $w \in dom(\mathcal{M})$ and valuations $V\}$

2 Tutorial Questions

- Show that there is no bisimulation between $\mathcal{M}, w$ and $\mathcal{M}', w'$.



$\mathcal{M}$ $\qquad\qquad$ $\mathcal{M}'$

- Find frames $\mathcal{F}_1 = \langle W_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle W_2, R_2 \rangle$ such that there is a modal formula $\varphi \in \mathcal{L}$ such that

$$\mathcal{F}_1 \models \varphi \qquad \text{and} \qquad \mathcal{F}_2 \not\models \varphi.$$

  Furthermore, find valuations V_1 and V_2 on $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively such that

$$(\mathcal{F}_1, V_1), w_1 \leftrightarrows (F_2, V_2), w_2$$

  for all $w_1 \in W_1$ and all $w_2 \in W_2$.

- We have seen that the universal modality $A\varphi$, where $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$, is not definable in the basic modal language. How do we modify the definition of bisimulation so that it preserves truth in a basic modal language with a a universal modality?

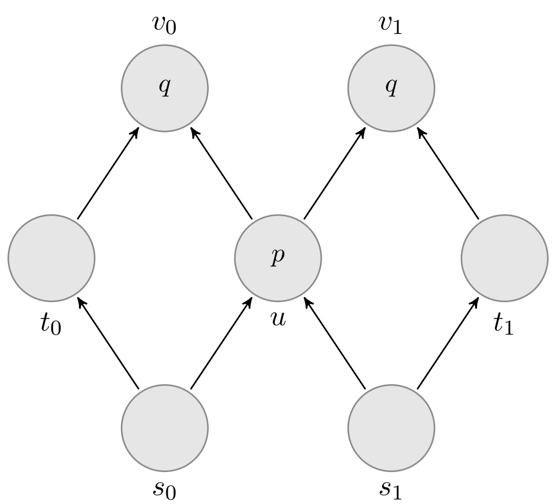
- Prove that the difference modality $D\varphi$ defined as $\mathcal{M}, w \models D\varphi$ iff there is a $v \in W$ such that $w \neq v$ and $\mathcal{M}, v \models \varphi$ is not definable in the basic modal language. Show that the universal modality is expressive in a language with the difference modality.

- The basic temporal language has two modalities: $F\varphi$ with the intended meaning "φ is true at some point in the future" and $P\varphi$ with the intended meaning "φ is true at some point in the past". This language can be interpreted on a model $\mathcal{M} = \langle W, R, V \rangle$. Use the **converse** of R, $R^{-1} = \{(v, w) \mid (w, v) \in R\}$, when interpreting the past modality. Truth for the basic temporal language is (I only give the definition for the modalities):

  - $\mathcal{M}, w \models F\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
  - $\mathcal{M}, w \models P\varphi$ iff for all $v \in W$, if $wR^{-1}v$, then $\mathcal{M}, v \models \varphi$

  Does bisimulation preserve truth for the basic temporal language? Hint: note that $\langle \mathbb{Z}, <, V \rangle, 0$ and $\langle \mathbb{N}, <, V \rangle, 0$ are bisimilar. How do you modify the definition of bisimulation so that it preserves truth for the temporal modal language?

- Show that the until operator $U(\varphi, \psi)$ with the intended meaning ψ is true until φ is true is not definable in the basic temporal language. The definition of the until operator is: $\mathcal{M}, w \models U(\varphi, \psi)$ iff there is a $v \in W$, wRv such that $\mathcal{M}, v \models \varphi$ and for all $u \in W$, if wRu and uRv, then $\mathcal{M}, u \models \psi$. Hint: consider the following model. Does s_0 satisfy $U(q, p)$? What about if the states s_1, t_1 and v_1 are removed?



3 Correspondence Theory

Definition 3.1 (Frame) A pair $\langle W, R\rangle$ with W a nonempty set of states and $R \subseteq W \times W$ is called a **frame**. Given a frame $\mathcal{F} = \langle W, R\rangle$, we say the model $\mathcal{M}$ **is based on the frame** $\mathcal{F} = \langle W, R\rangle$ if $\mathcal{M} = \langle W, R, V\rangle$ for some valuation function V. $\triangleleft$

Definition 3.2 (Frame Validity) Given a frame $\mathcal{F} = \langle W, R\rangle$, a modal formula φ is **valid on** $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, provided $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ based on $\mathcal{F}$. $\triangleleft$

Suppose that P is a property of relations (eg., reflexivity or transitivity). We say a frame $\mathcal{F} = \langle W, R\rangle$ has property P provided R has property P. For example,

- $\mathcal{F} = \langle W, R\rangle$ is called a **reflexive frame** provided R is reflexive, i.e., for all $w \in W$, wRw.
- $\mathcal{F} = \langle W, R\rangle$ is called a **transitive frame** provided R is transitive, i.e., for all $w, x, v \in W$, if wRx and xRv then wRv.

Definition 3.3 (Defining a Class of Frames) A modal formula φ **defines the class of frames with property** P provided for all frames $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ has property P. $\triangleleft$

Remark 3.4 *Note that if $\mathcal{F} \models \varphi$ where φ is some modal formula, then $\mathcal{F} \models \varphi^*$ where φ^* is any* **substitution instance** *of φ. That is, φ^* is obtained by replacing sentence letters in φ with modal formulas. In particular, this means, for example, that in order to show that $\mathcal{F} \not\models \Box\varphi \to \varphi$ it is enough to show that $\mathcal{F} \not\models \Box p \to p$ where p is a sentence letter. (This will be used in the proofs below).*

Fact 3.5 *$\Box\varphi \to \varphi$ defines the class of reflexive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \to \varphi$ iff $\mathcal{F}$ is reflexive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is reflexive and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \to \varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. Then for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$. Since R is reflexive, we have wRw. Hence, $\mathcal{M}, w \models \varphi$. Therefore, $\mathcal{M}, w \models \Box\varphi \to \varphi$, as desired.

($\Rightarrow$) We argue by contraposition. Suppose that $\mathcal{F}$ is not reflexive. We must show $\mathcal{F} \not\models \Box\varphi \to \varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \to p$ for some sentence letter p. Since $\mathcal{F}$ is not reflexive, there is a state $w \in W$ such that it is not the case that wRw. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(p) = \{v \mid v \neq w\}$. Then $\mathcal{M}, w \models \Box p$ since, by assumption, for all $v \in W$ if wRv, then $v \neq w$ and so $v \in V(p)$. Also, notice that by the definition of V, $\mathcal{M}, w \not\models p$. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg p$, and so, $\mathcal{F} \not\models \Box p \to p$.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Box\varphi \to \varphi$. We must show that for all x if xRx. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid xRu\}$. Since $\Box p$ is true at x we also have p true at x. This means that $x \in V(p)$, hence, xRx. QED

Fact 3.6 $\Box\varphi \to \Box\Box\varphi$ *defines the class of transitive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \to \Box\Box\varphi$ iff $\mathcal{F}$ is transitive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is transitive and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \to \Box\Box\varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. We must show $\mathcal{M}, w \models \Box\Box\varphi$. Suppose that $v \in W$ and wRv. We must show $\mathcal{M}, v \models \Box\varphi$. To that end, let $x \in W$ be any state with vRx. Since R is transitive and wRv and vRx, we have wRx. Since $\mathcal{M}, w \models \Box\varphi$, we have $\mathcal{M}, x \models \varphi$. Therefore, since x is an arbitrary state accessible from v, $\mathcal{M}, v \models \Box\varphi$. Hence, $\mathcal{M}, w \models \Box\Box\varphi$, and so, $\mathcal{M}, w \models \Box\varphi \to \Box\Box\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ is not transitive. We must show $\mathcal{F} \not\models \Box\varphi \to \Box\Box\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \to \Box\Box p$ for some sentence letter p. Since $\mathcal{F}$ is not transitive, there are states $w, v, x \in W$ with wRv and vRx but it is not the case that wRx. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(p) = \{y \mid y \neq x\}$. Since $\mathcal{M}, x \not\models p$ and wRv and vRx, we have $\mathcal{M}, w \not\models \Box\Box p$. Furthermore, $\mathcal{M}, w \models \Box p$ since the only state where p is false is x and it is assumed that it is not the case that wRx. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg\Box\Box p$, and so, $\mathcal{F} \not\models \Box p \to \Box\Box p$, as desired.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Box\varphi \to \Box\Box\varphi$. We must show that for all x, y, z if xRy and yRz then xRz. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid xRu\}$. Since $\Box p$ is true at x we also have $\Box\Box p$ true at x. This means that for all y if xRy then (for all z if yRz we have $z \in V(p)$). Recall that $z \in V(p)$ means that xRz. Putting everything together we have: for all y if xRy then for all z if yRz then xRz. QED

Fact 3.7 $\varphi \to \Box\Diamond\varphi$ *defines the class of symmetric frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \varphi \to \Box\Diamond\varphi$ iff $\mathcal{F}$ is symmetric.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is symmetric and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \varphi \to \Box\Diamond\varphi$. Suppose that $\mathcal{M}, w \models \varphi$. We must show $\mathcal{M}, w \models \Box\Diamond\varphi$. Suppose that $v \in W$ and wRv. We must show $\mathcal{M}, v \models \Diamond\varphi$. Since R is symmetric and wRv, we have vRw. Since $\mathcal{M}, w \models \varphi$, we have $\mathcal{M}, v \models \Diamond\varphi$. Hence, $\mathcal{M}, w \models \Box\Diamond\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ is not symmetric. We must show $\mathcal{F} \not\models \varphi \to \Box\Diamond\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models p \to \Box\Diamond p$ for some sentence letter p. Since $\mathcal{F}$ is not symmetric, there are states $w, v \in W$ with wRv but it is not the case that vRw. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(p) = \{w\}$. Then, $\mathcal{M}, w \models p$. Since it is not the case that vRw and w is the only state satisfying p, we have $\mathcal{M}, v \not\models \Diamond p$. This means that $\mathcal{M}, w \not\models \Box\Diamond p$ (since wRv and $\mathcal{M}, v \not\models \Diamond p$).

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \varphi \to \Box\Diamond\varphi$. We must show that for all x, y if xRy then yRx. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid u = x\}$. Since p is true at x we also have $\Box\Diamond p$ true at x. This means that for all y if xRy then there is a z such that yRz and $z \in V(p)$. Recall that $z \in V(p)$ means that $z = x$. Putting everything together we have: for all y if xRy then there is a z such that z if yRz then $x = z$. This property is symmetry. QED

Fact 3.8 $\Diamond\Box\varphi \to \Box\Diamond\varphi$ *defines the confluence property: for all x, y, z if xRy and xRz then there is a s such that yRs and zRs.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$ iff $\mathcal{F}$ satisfies the confluence property: for all x, y, z if xRy and xRz then there is a s such that yRs and zRs.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ satisfies confluence and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Diamond\Box\varphi \to \Box\Diamond\varphi$. Suppose that $\mathcal{M}, w \models \Diamond\Box\varphi$. We must show $\mathcal{M}, w \models \Box\Diamond\varphi$. Suppose that $x \in W$ with wRx. Since $\mathcal{M}, w \models \Diamond\Box\varphi$, there is a y such that wRy and $\mathcal{M}, y \models \Box\varphi$. Since wRx and wRy, by the confluence property, there is a $s \in W$ with xRs and yRs. Since yRs and $\mathcal{M}, y \models \Diamond\varphi$, we have $\mathcal{M}, s \models \varphi$. Then, since xRs, we have $\mathcal{M}, x \models \Diamond\varphi$. Hence, $\mathcal{M}, w \models \Box\Diamond\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ does not satisfy confluence. We must show $\mathcal{F} \not\models \Diamond\Box\varphi \to \Box\Diamond\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Diamond\Box p \to \Box\Diamond p$ for some sentence letter p. Since $\mathcal{F}$ does not satisfy confluence, there are states $w, x, y \in W$ with wRx and wRy but there is no s such that xRs and yRs. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(p) = \{v \mid yRv\}$. Then, $\mathcal{M}, y \models \Box p$ (since all states accessible from y satisfy p). Since there is no s such that xRs and yRs, we also have $\mathcal{M}, x \not\models \Diamond p$. Since wRx and wRy, we have $\mathcal{M}, w \not\models \Box\Diamond p$ and $\mathcal{M}, w \models \Diamond\Box p$. Hence, $\Diamond\Box p \to \Box\Diamond p$ is not valid.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$. We must show that for all x, y, z if xRy and xRz, then there is a s such that yRs and zRs. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid yRu\}$. Let y, z be states with xRy and xRz. Since, $\mathcal{M}, y \models \Box p$, we have $\mathcal{M}, x \models \Diamond\Box p$. This means that $\mathcal{M}, x \models \Box\Diamond p$. Hence, since xRz, we have $\mathcal{M}, z \models \Diamond p$. Thus, there is a states v such that zRv and $v \in V(p)$. Since $v \in V(p)$, we have yRv. Putting everything together we have: for all x, y, z if xRy and xRz, then there is a s such that yRs and zRs. QED

Not all modal formulas correspond to first-order properties:

Basic properties of first-order logic:

- **Compactness**: Γ is satisfiable iff every finite subset is satisfiable.
- **Löwenheim-Skolem Theorem**: If Γ is satisfiable, then it is satisfiable on a countable model.

Fact 3.9 $\mathcal{F} \models \Box(\Box\varphi \to \varphi) \to \Box\varphi$ *iff $\mathcal{F}$ is transitive and converse well-founded.*

Fact 3.10 $\Box\Diamond\varphi \to \Diamond\Box\varphi$ *does not correspond to a first-order condition.*

Theorem 3.11 (Goldblatt-Thomason) *A first-order definable class K of frames is modally definable iff it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

Sahlqvist's Algorithm (see section 9.3 of *Modal Logic for Open Minds* and Sections 3.5 - 3.7 of *Modal Logic* by Blackburn, de Rijke and Venema for an extensive discussion).

Standard Translation

First-order language with an appropriate signature

$st_x : \mathcal{L} \to \mathcal{L}_1$

$$
\begin{aligned}
st_x(p) &= Px \\
st_x(\neg\varphi) &= \neg st_x(\varphi) \\
st_x(\varphi \wedge \psi) &= st_x(\varphi) \wedge st_x(\psi) \\
st_x(\Box\varphi) &= \forall y(xRy \to st_y(\varphi)) \\
st_x(\Diamond\varphi) &= \exists y(xRy \wedge st_y(\varphi))
\end{aligned}
$$

$st_y : \mathcal{L} \to \mathcal{L}_1$

$$
\begin{aligned}
st_y(p) &= Py \\
st_y(\neg\varphi) &= \neg st_y(\varphi) \\
st_y(\varphi \wedge \psi) &= st_y(\varphi) \wedge st_x(\psi) \\
st_y(\Box\varphi) &= \forall x(yRx \to st_x(\varphi)) \\
st_y(\Diamond\varphi) &= \exists x(yRx \wedge st_x(\varphi))
\end{aligned}
$$

Fact: Modal logic falls in the two-variable fragment of $\mathcal{L}_1$.

Lemma For each $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M} \Vdash st_x(\varphi)[x/w]$.

Lemma $\mathcal{F} \models \varphi$ iff $\mathcal{F} \Vdash \forall P_1 \forall P_2 \cdots \forall P_n \forall x\ st_x(\varphi)$
where the P_i correspond to the atomic propositions p_i in φ.