# Lecture 6: Incompleteness

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## 1 Tutorial Questions

**General Frames**: A general frame is a tuple $\langle W, R, A\rangle$ where $A$ is a collection of *admissible* subsets of $W$ closed under the following operations:

- If $X, Y \in A$ then $X \cap Y \in A$
- If $X \in A$ then $\overline{X} \in A$
- If $X \in A$ then $l(X) = \{w \mid \text{for all } v \text{ if } wRv \text{ then } v \in X\} \in A$.

A model based on a general frame $\mathcal{F} = \langle W, R, A\rangle$ is a tuple $\mathcal{M} = \langle W, R, A, V\rangle$ where for each $p \in \mathsf{At}$, $V(p) \in A$.

- Consider the frame $\langle \mathbb{N}, <\rangle$. Prove that the McKinsey formula $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$ is not valid on this frame.
- Consider the general frame $\langle \mathbb{N}, <, A\rangle$, where $A$ is the set of all finite or co-finite subsets of $\mathbb{N}$. Prove that the McKinsey formula is valid on this general frame.

Convince yourself that the following are true:

- $\mathcal{F} \models (\Diamond\varphi \wedge \Diamond\psi) \rightarrow (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\Diamond\varphi \wedge \psi))$ iff $\mathcal{F}$ non-branching to the right (for all $w, v, x$ if $wRv$ and $wRx$ then either $vRx$ or $xRv$ or $v = x$).
- $\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi$ iff $\mathcal{F}$ is unbounded (to the right: for all $w$ there is a $v$ such that $wRv$).
- $\mathcal{F} \models \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$ iff $\mathcal{F}$ is transitive and converse well-founded.

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**Temporal Logic**: Let $\mathcal{M} = \langle T, R, V \rangle$ be a Kripke model.

- $\mathcal{M}, t \models F\varphi$ iff there exists a $t'$ such that $tRt'$ and $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models P\varphi$ iff there exists a $t'$ such that $t'Rt$ and $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models G\varphi$ iff for all $t'$, if $tRt'$ then $\mathcal{M}, t' \models \varphi$
- $\mathcal{M}, t \models H\varphi$ iff for all $t'$, if $t'Rt$ then $\mathcal{M}, t' \models \varphi$

The minimal temporal logic $\mathbf{K_t}$ contains the following axiom schemes and rules:

- Propositional logic
- $G(\varphi \to \psi) \to (G\varphi \to G\psi)$
- $G(\varphi \to \psi) \to (G\varphi \to G\psi)$
- $\varphi \to GP\varphi$
- $\varphi \to HF\varphi$
- From $\varphi$ derive $\bigcirc\varphi$ where $\bigcirc = G, H$
- Modus Ponens

Let $\mathbf{K_t Tho}$ be the temporal logic extending $\mathbf{K_t}$ with the axiom schemes

- $Fp \wedge Fq \to (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q))$
- $Gp \to Fp$
- $H(Hp \to p) \to Hp$

**Fact 1** $\mathbf{K_t Tho}$ *is consistent.*

**Fact 2** *If $\mathcal{F} = \langle T, R \rangle$ is a frame for* $\mathbf{K_t Tho}$*, then for $t \in T$, $\{u \mid tRu\}$ is an unbounded strict total order.*

**Fact 3** *If $\mathcal{F} = \langle T, R \rangle$ is a frame for* $\mathbf{K_t Tho}$*, then $\mathcal{F} \not\models GFp \to FGp$.*

**Fact 4** *The logic* $\mathbf{K_t ThoM}$ *which extends* $\mathbf{K_t Tho}$ *with the axiom scheme $GF\varphi \to FG\varphi$ is* consistent *and* incomplete *(I.e.,* $\mathbf{K_t ThoM}$ *is not the logic for any class of frames).*