Neighborhood Semantics for Modal Logic

Lecture 1

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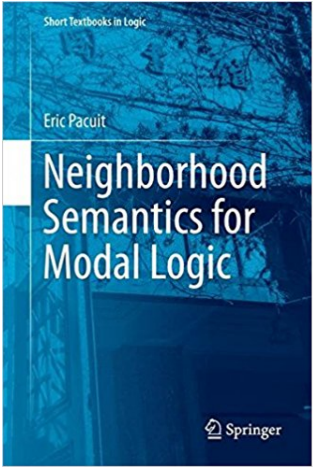
Text


Short Textbooks in Logic
Eric Pacuit
Neighborhood Semantics for Modal Logic
Springer

Text

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Non-normal modal logics

Non-normal modal logics

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

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(Non-)Normal Modal Logic

Let $\mathcal{L}$ be the basic modal language.

A **modal logic** is a set of formulas from $\mathcal{L}$. If $\mathbf{L}$ is a modal logic, then we write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

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A modal logic **L** is **normal** provided **L** is

- contains propositional logic (i.e., all instances of the propositional axioms and closed under Modus Ponens)
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- contains all instances of *K* ($\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$); and
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Normal Modal Logic

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

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Theorem. K is sound and strongly complete with respect to the class of all relational frames.

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Theorem. For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

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Theorem. **K** $+ \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive relational frames.

Are there non-normal extensions of **K**?

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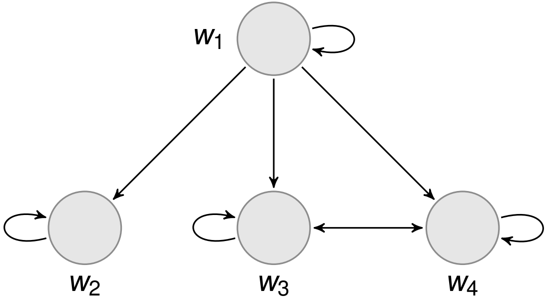
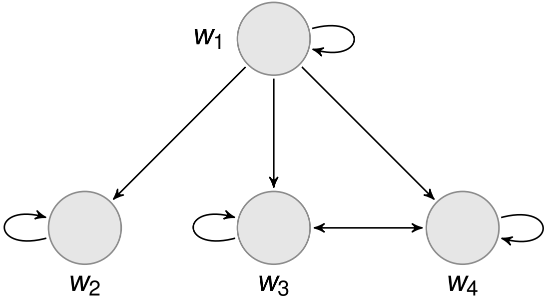
Let **L** be the smallest modal logic containing

- **S4** (**K** $+\ \Box\varphi \to \varphi\ +\ \Box\varphi \to \Box\Box\varphi$)
- all instances of *M*: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4.**



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$$



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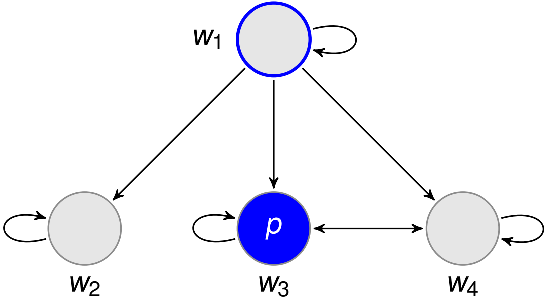

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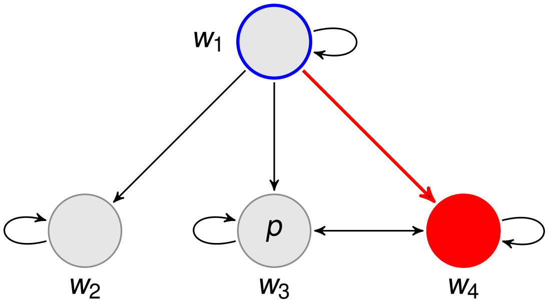

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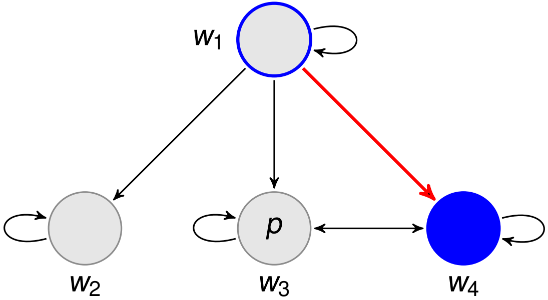

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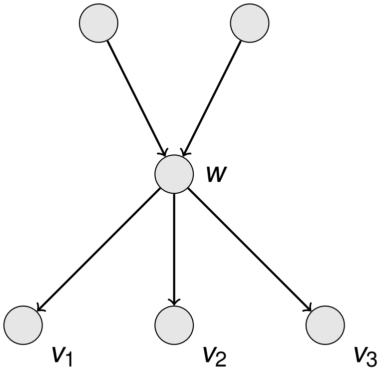
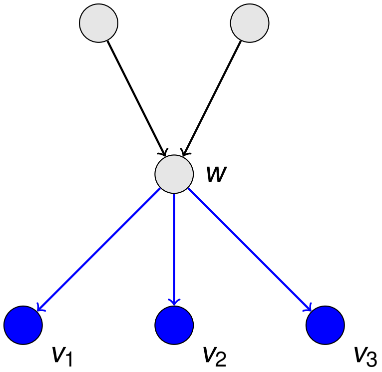
✓ Non-normal modal logics

1. Neighborhood semantics for modal logic


w
v_1
v_2
v_3



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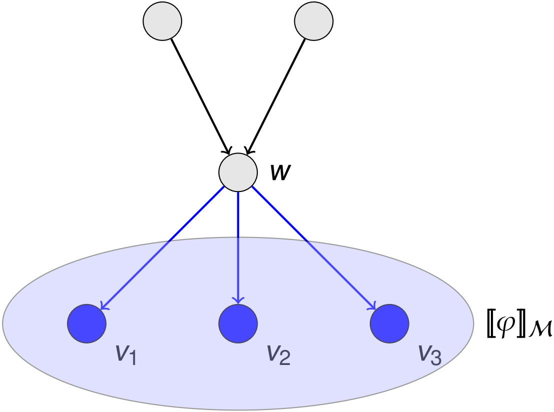
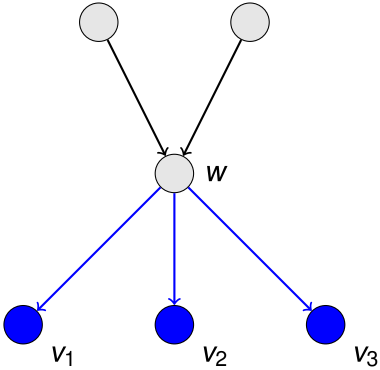


$\mathcal{M}, w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

...**the** neighborhood of w is **contained in** the truth-set of φ



$\mathcal{M}, w \models \boxplus\varphi$ iff $R(w) = [\![\varphi]\!]_{\mathcal{M}}$

...**the** neighborhood of w **is** the truth-set of φ

Neighborhoods in Topology

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

J. Dugundji. *Topology*. 1966.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

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What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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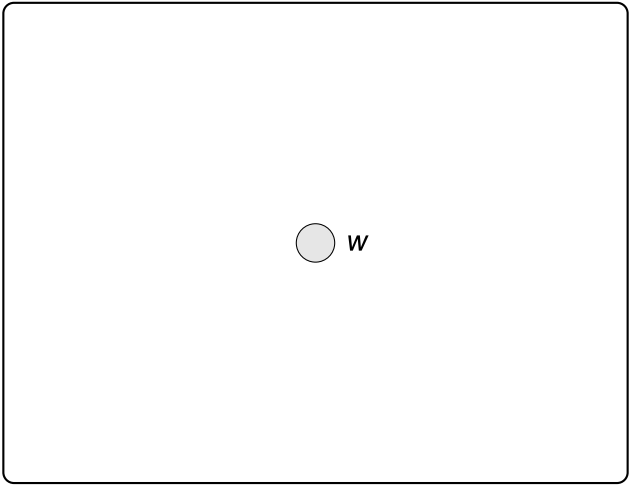
an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.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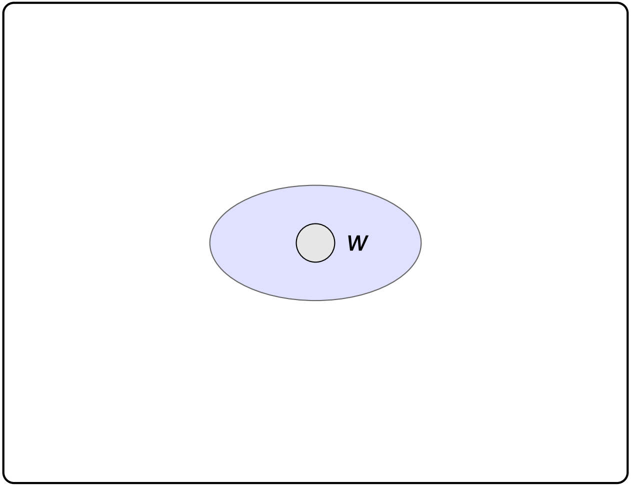
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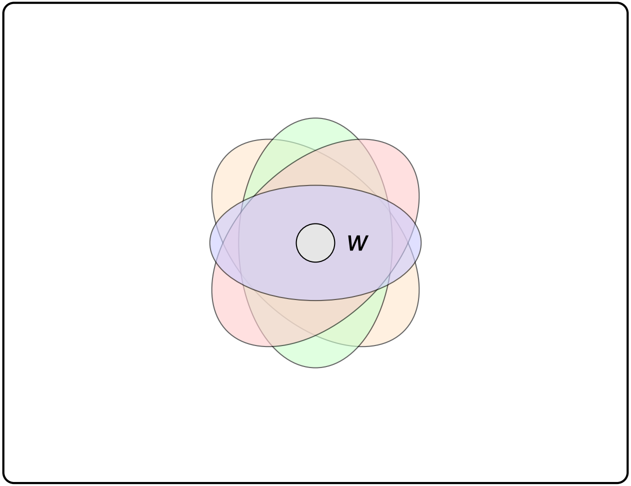
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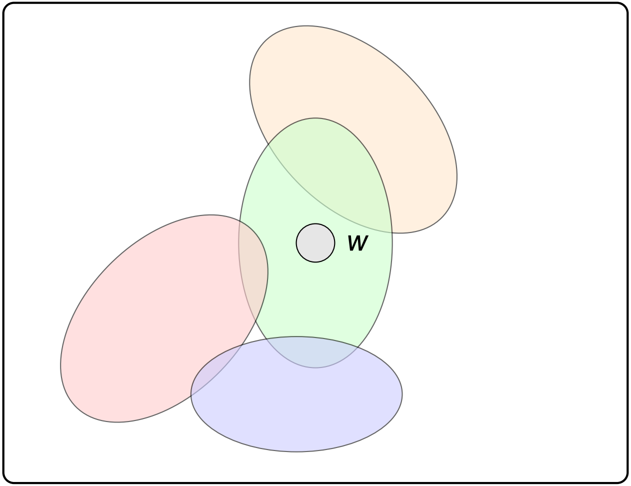
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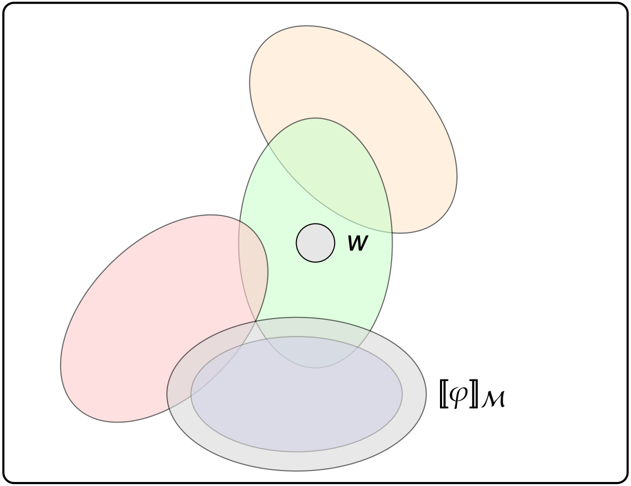

$\mathcal{M}, w \models \Box\varphi$ iff **there is** a neighborhood of w **contained in** $[\![\varphi]\!]_{\mathcal{M}}$

Relational model: $\langle W, R, V\rangle$ where $R : W \to \wp(W)$

$w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]$

Neighborhood model: $\langle W, N, V\rangle$ where $N : W \to \wp(\wp(W))$

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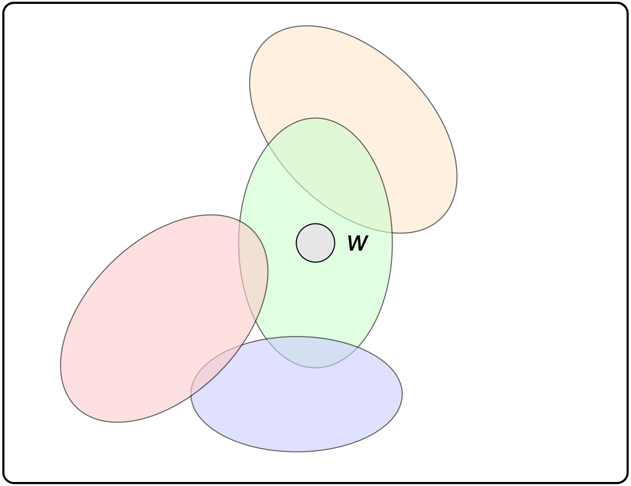

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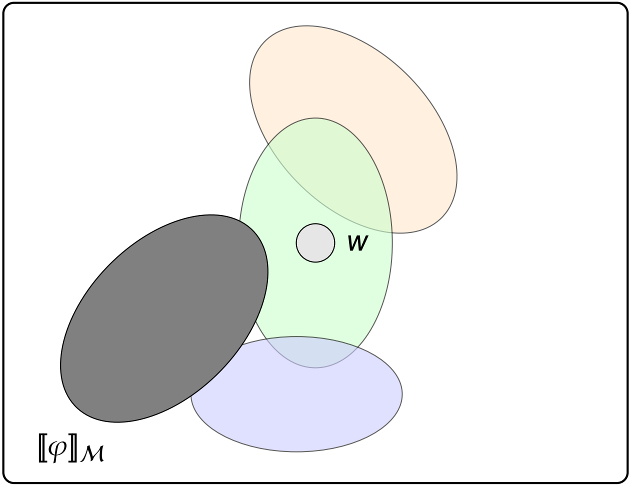

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✓ Non-normal modal logics

✓ Neighborhood semantics for modal logic

Why non-normal modal logic?

Why neighborhood models?

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

(Montague, pg. 73)

R. Montague. *Pragmatics and Intentional Logic*. 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

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K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

> *This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.*
>
> (pg. 1)

Two routes to a logical framework

1. Identify interesting patterns that you (do not) want to represent
2. Identify interesting structures that you want to reason about

- Logical omniscience
- Logics of knowledge and beliefs
- Logic of high probability
- Logics of ability
- Deontic logics
- Logics of classical deduction
- Logics of group decision making

Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
Nec	From φ, infer $\Box\varphi$
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$ *closure under logical implication*
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Logical Omniscience/Knowledge Closure

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K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *closure under known implication*
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Logical Omniscience/Knowledge Closure

W. Holliday. *Epistemic closure and epistemic logic I: Relevant alternatives and subjunctivism*. Journal of Philosophical Logic, 1 - 62, 2014.

J. Halpern and R. Puccella. *Dealing with logical omniscience: Expressiveness and pragmatics*. Artificial Intelligence 175(1), pgs. 220 - 235, 2011.

Logics of High Probability

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

R. Stalnaker. *On logics of knowledge and belief*. Philosophical Studies 128, 169 199, 2006.

(K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$
(T)	$K\varphi \rightarrow \varphi$
(4)	$K\varphi \rightarrow KK\varphi$
(Nec)	From φ infer $K\varphi$

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(T)	$K\varphi \rightarrow \varphi$
(4)	$K\varphi \rightarrow KK\varphi$
(Nec)	From φ infer $K\varphi$

(PI)	$B\varphi \rightarrow KB\varphi$
(NI)	$\neg B\varphi \rightarrow K\neg B\varphi$
(KB)	$K\varphi \rightarrow B\varphi$
(D)	$B\varphi \rightarrow \langle B\rangle\varphi$
(SB)	$B\varphi \rightarrow BK\varphi$

$$(.2) \qquad \langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi$$

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What happens if we drop axiom (4)?

$$(.2) \qquad \langle K \rangle K\varphi \rightarrow K \langle K \rangle \varphi$$

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Claim. B is a normal modal operator.

What happens if we drop axiom (4)?

Under certain conditions, B is not a normal modal operator.

D. Klein, N. Gratzl, and O. Roy. *Introspection, normality and agglomeration*. Logic, Rationality, and Interaction, 5th Workshop, LORI 2015, 195 206.

Logic of Deduction

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Interpretation: $(\cdot)^* : \mathsf{At} \to \wp(\Sigma)$

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P. Naumov. *On modal logic of deductive closure*. APAL (2005).

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Fact: $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ is not valid.

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(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

~~(C)~~ $\cancel{\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)}$

~~(N)~~ $\cancel{\Box\top}$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

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(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

$$\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

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1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

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✓ If Jones murders Smith, then Jones ought to murder Smith gently

4. Jones ought to murder Smith gently

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✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith

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Why non-normal modal logic? ✓

Why neighborhood models?

- Subset spaces, neighborhood frames/models, reasoning about subset spaces
- Interesting mathematical structures: Ultrafilters, topologies, hypergraphs
- Logic of knowledge, evidence and belief
- Coalitional logic

Some Terminology: Subset Spaces

Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cap_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under unions if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under complements if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

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Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ contains the unit provided $W \in \mathcal{F}$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
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A few more definitions

- $\mathcal{F}$ is a filter if $\mathcal{F}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.
- $\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{F}$ or $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is a topology if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains its core and is supplemented.

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \rightarrow \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N\rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N\rangle$ is a tuple $\langle W, N, V\rangle$ where $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathrm{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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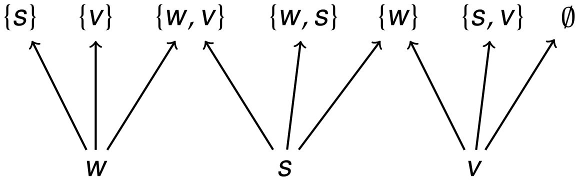

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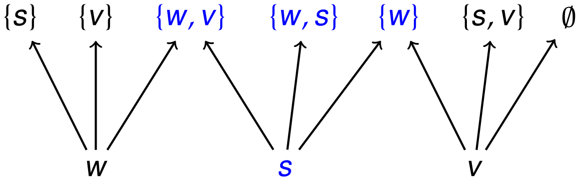

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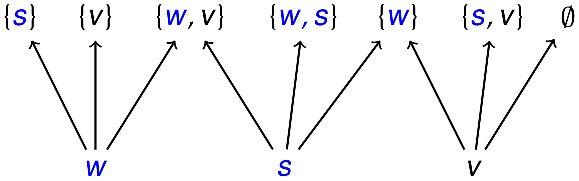

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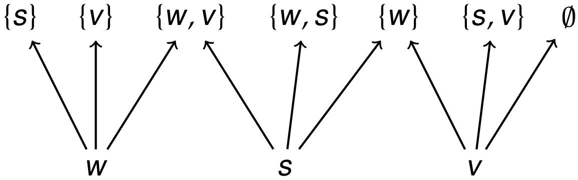

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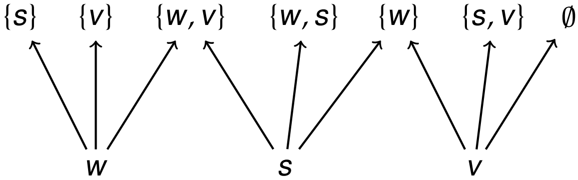

$$\mathfrak{M}, s \models \Box p$$

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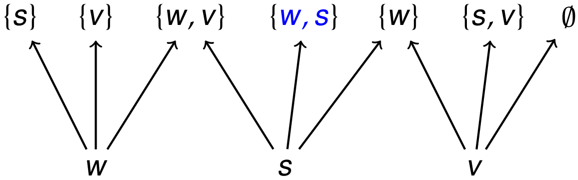

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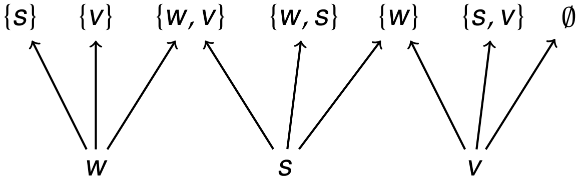

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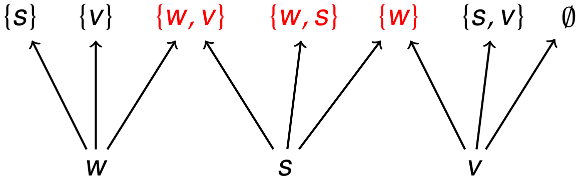

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$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

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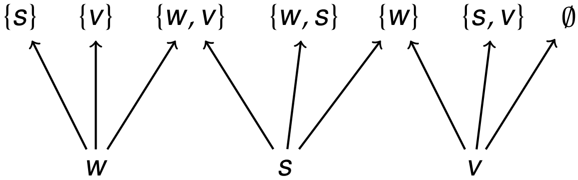

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Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$

$$\begin{tikzcd}[column sep=small] \{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\ & w \arrow[ul] \arrow[u] \arrow[ur] & & s \arrow[ul] \arrow[u] \arrow[ur] & & v \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

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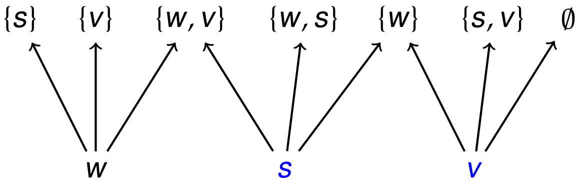

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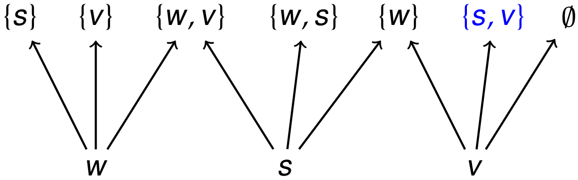

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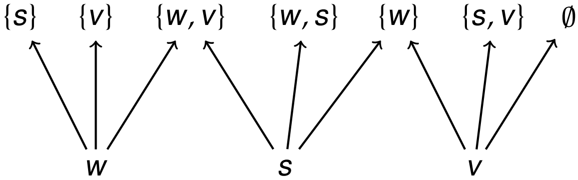

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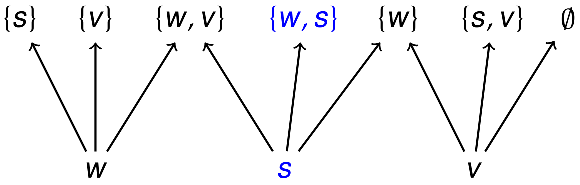

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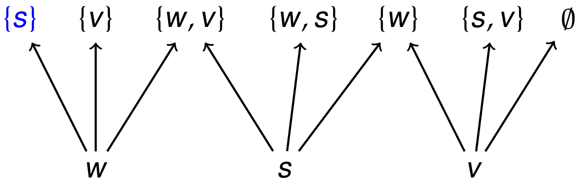

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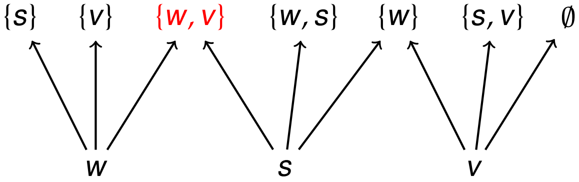

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Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X, \mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X, \mathfrak{M}, v \models \varphi$

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Other modal operators

- $\mathfrak{M}, w \models \langle\,]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\, \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Other modal operators

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Lemma

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

- $\mathfrak{M}, w \models \langle\,]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\, \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Lemma

1. *If $\varphi \rightarrow \psi$ is valid in $\mathfrak{M}$, then so is $\langle\,]\varphi \rightarrow \langle\,]\psi$.*
2. $\langle\,](\varphi \wedge \psi) \rightarrow (\langle\,]\varphi \wedge \langle\,]\psi)$ *is valid in* $\mathfrak{M}$

Investigate analogous results for the other modal operators defined above.

Non-normal modal logics

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Non-normal modal logics

~~(M)~~	$\cancel{\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi}$
~~(C)~~	$\cancel{\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)}$
~~(N)~~	$\cancel{\Box\top}$
~~(K)~~	$\cancel{\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)}$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
~~(Nec)~~	~~from $\vdash \varphi$ infer $\vdash \Box\varphi$~~
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

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RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

E is the smallest classical modal logic.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

In **E**, *M* is equivalent to

(*Mon*) $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

PC 6. Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

PC	Propositional Calculus
E	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
Mon	$\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$
C	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$
N	$\Box\top$
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
RE	$\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$
Nec	$\dfrac{\varphi}{\Box\varphi}$
MP	$\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

K is the smallest normal modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

K $=$ **EMCN**

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

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K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

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E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \mathsf{At} \to \wp(W)$ is a valuation function.

Why non-normal modal logic?

Why neighborhood models?