Finite Models Constructed From Canonical Formulas

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Abstract

This paper obtains the weak completeness and decidability results for standard systems of modal logic using models built from formulas themselves. This line of work began with Fine [4]. There are two ways in which our work advances on that paper: First, the definition of our models is mainly based on the relation Kozen and Parikh used in their proof of the completeness of PDL, see [7]. The point is to develop a general model-construction method based on this definition. We do this and thereby obtain the completeness of most of the standard modal systems, and in addition apply the method to some other systems of interest. None of the results use filtration, but in our final section we explore the connection.

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1 Introduction

The *normal forms* of propositional modal logic have been discovered several times. These are the analog of the Scott sentences in modal logic, and they also are generalizations of state descriptions from propositional logic. We'll define them in due course, but here are some examples:

$$
\begin{array}{lcl}
\alpha & = & \neg p \wedge \neg q \wedge \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Box(((p \wedge q)) \vee (\neg p \wedge q)) \\
\beta & = & p \wedge \neg q \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Box(((\neg p \wedge q)) \vee (\neg p \wedge q)) \\
\chi & = & \neg p \wedge q \wedge \Diamond\alpha \wedge \Diamond\beta \wedge \Box(\alpha \vee \beta)
\end{array}
$$

The primary source on the use of normal forms is Kit Fine's paper 1975 paper "Normal forms in modal logic" [4]. Presumably Fine called them "normal forms" because every modal formula is equivalent to a disjunction of a finite set of them. In a different way, such sentences serve as *characterizing sentences* (or approximations to such sentences). This means that the bisimulation type of a given model-world pair is an infinitary sentence built in the manner of the examples above; see [1], Theorem 11.12. This result will not be important to us, and indeed we shall refer to these formulas as *canonical formulas* in our development.

Fine [4] claim that "Normal forms have been comparatively neglected in the study of modal sentential logic" seems even more cogent thirty years after its publication. The topic is missing from most recent textbooks, and only a handful of papers discuss it. There are several possible reasons for this. First, normal forms give weak completeness and decidability results, and these can be obtained as well via the method of filtration, as first shown by Lemmon and Scott. So one might reasonably ask what the advantage of normal form proofs could be. This is answered by Fine's claim that normal form methods are more elegant. Indeed, as David Makinson's review [9] points out, "[Normal forms are] applied with flair and elegance to the modal logics $K, T, K4$, and a fairly broad class of 'uniform' modal logics. In the case of K the construction turns out to be quite simple; in the other cases it is rather intricate."

And this brings us to the second possible reason for the neglect of normal forms. There has not been an account of what the method consists of that allows us to ask what it can and cannot do. Thus the original applications in [4] seem in retrospect to be ad hoc. To be more specific on this point, Fine's main construction builds finite Kripke models from the normal forms themselves. The "intricate" constructions boil down to the specification of a particular accessibility relation on a particular set of normal forms of a given (finite) height and over a given (finite) set of atomic propositions. The original definitions of the subset and the relation are indeed special, and it would appear that they must be tailored logic-by-logic.

This paper attempts to re-open the matter of building Kripke models from the formulas of the logic itself. It develops the topic from scratch in Sections 2 and 3 and then turns to new applications. Our re-working of the topic aims to develop it as a method in the sense that we settle on one main construction, the models $\mathbb{C}_{h,n}(L)$ introduced in Section 4. This relates to our point just above. Our definitions are arguably simpler and more ‘canonical’. We have $\mathbb{C}_{h,n}(L) = (\mathcal{C}_{h,n}(L), \xrightarrow{c}, v)$, where

1. $C_{h,n}(L)$ is a certain set of formulas of modal height $\leq h$ built from the first n atomic propositions, all of which are consistent in the logic L.

2. $\alpha \xrightarrow{c} \beta$ iff $\alpha \wedge \Diamond\beta$ is consistent in L.

3. $v(p_i) = \{\alpha : \vdash \alpha \to p_i\}$.

Point (3) is what one would expect from any model construction where the worlds are formulas. The important point is (2). This definition comes from the Kozen-Parikh [7] proof of the completeness of propositional dynamic logic. That paper was published six years after [4]. It is tempting to think that this paper is the version of Fine’s [4] that comes with the hindsight of the main definition of [7]. (And in the other direction, we use the normal form proof to simplify the work of [7] a bit, since we bypass filtration.)

This paper is mainly a study of the models $\mathbb{C}_{h,n}(L)$, and applications of them in proving weak modal completeness theorems. We also ask about the relation of our work to filtration in Section 7.1. Before turning to the specific contributions of the paper, we should emphasize that our work only gives the *weak* completeness results. (That is, we prove that consistent sentences in various logics have models of certain types; but our work does not directly carry over to show that consistent *sets* of sentences have models of the appropriate types.) To get strong completeness results from our work, the easiest way would seem to be via semantic compactness theorems, provable using ultraproducts.

Specific completeness results Section 5 contains the weak completeness results for all modal logics built from K using T, B, D, 4 and 5 with respect to the expected classes of finite models. The only exceptions are $K5$ and $KD5$; for those the methods do not work. As we have mentioned, the completeness results in [4] are for K, KT, KD, $K4$, and the uniform logics such as KM; also mentioned at the end are KB and $S4$. For $S4$, the method gives weak completeness for finite preorders. Sections 5 and 6 also contain completeness of the provability logic KL, $K4McK$, and the logic $K\Box^*$ of the transitive closure operator. We remind the reader that filtration is not used in any of our arguments. (Also, we have nothing to say about strong completeness.) We believe that our development of the weak completeness results is somewhat simpler than the standard approach. On a related pedagogical point, we think that pictures of the models $\mathcal{C}_{h,n}(L)$ for various logics L in Figure 1 should help students who prefer to have presentations which are as concrete as possible.

How to read this paper This paper will read differently depending on what the reader brings. It is mainly written for those with no experience with modal completeness proofs, and indeed the paper itself can be used as a treatment of the central results in the area that we think is faster and easier than more popular methods. However, all of the specific completeness results in this paper are already known. Most appear in standard textbooks, such as Blackburn, de Rijke, and Venema [3]. So readers who know those results might well wonder what the novelty is and whether the re-working of old results is a reasonable thing to do in the first place. Those readers might prefer to read or skim the paper until the end of Section 4, and then take up Section 7.1.

The technical material in this paper is quite elementary. Most of it could be read by anyone who knows the completeness of classical propositional logic in any logical system, the Kripke semantics of modal logic, and the specific modal systems such as K, $S4$, etc.

History My interest in these matters goes back to work with Jon Barwise on characterization results for infinitary modal logic, and later applications of the same construction to the modal correspondence theory. (See [1], Theorem 11.12, and also [2].) Analogs of the same construction for finitary and infinitary modal logics were the leading idea behind coalgebraic logic [10]. However, in none of these works does one find a model construction based on the characterizing formulas. Later, while teaching modal logic to undergraduates, I was faced with the task of teaching completeness theorems to students who lacked the mathematics background to understand the traditional completeness-via-filtration arguments. So I worked out proofs using the characterizing formulas themselves. Since I had seen the Kozen-Parikh work on PDL, it was natural to adapt the idea. In writing up that work, I found that Fine had done the same thing in 1975. His work is not so well-known, I think: none of the readers of any of the papers mentioned above ever mentioned it to me. There have not been many papers that build on it. (One exception is Ghilardi [5], but its approach seems very different from this paper's. For that matter, a construction related to Fine's in the intuitionistic setting may be found in de Jongh's dissertation [6]. This predates Fine's paper.) And in looking at Fine's paper [4], there are some differences mainly due to the way that the models are defined. In any case, one of the purposes of this paper is to stimulate some new thinking about the whole matter of constructing finite models in modal logics using formulas themselves as worlds and with certain special relations as the accessibility, as we have mentioned above.

A didactic point One of our goals is to present weak completeness proofs in as simple a manner as possible. I believe that the approach here might be simpler than the standard one. The reason is that one gets by without Zorn's Lemma or quotients. To be fair, there are still some complexities: students have to be good with induction to work through the proofs. To use this material in a classroom setting would mainly mean presenting some of the results in detail while keeping others as exercises. I have found that this works, but my sample is too small to make a strong claim that the method works for students who find the standard approach tough going.

2 The $\oplus$ notation for "exactly one"

In this section, we introduce some notation that will be used throughout this paper. We always work with a countable set of *atomic propositions* $p_1, p_2, \ldots, p_n, \ldots$. We write $\oplus(\psi_1, \ldots, \psi_n)$ to mean that *exactly one* of $\psi_1, \ldots, \psi_n$ holds:

$$\bigvee_i (\psi_i \ \wedge \ \neg \bigvee_{j \neq i} \psi_j).$$

We also use this notation a bit sloppily when the list of formulas comes without a definite order, as in $\oplus\{\psi_1, \ldots, \psi_n\}$. For example, let

$$\mathsf{SD}_n = \text{the set of all state descriptions of order } n. \tag{1}$$

This is the set of formulas of the form $q_1 \wedge \ldots \wedge q_n$, with each q_i equal to either the corresponding atomic proposition p_i or its negation $\neg p_i$.

Lemma 2.1. *In any complete logical system for propositional logic* $\vdash \oplus \mathsf{SD}_n$.

Proof We use completeness and the semantic fact that $\models \oplus \mathsf{SD}_n$. $\dashv$

Lemma 2.2. *The following are equivalent in propositional logic:*

1. $\oplus(\varphi_1, \varphi_2, \ldots, \varphi_k) \wedge \oplus(\psi_1, \ldots, \psi_l)$
2. $\oplus\{\varphi_i \wedge \psi_j : 1 \leq i \leq k, 1 \leq j \leq l\}$

Lemma 2.3. *The following are equivalent in propositional logic:*

1. $\oplus(\varphi_1, \psi_1) \wedge \cdots \wedge \oplus(\varphi_n, \psi_n)$
2. $\oplus\{\bigwedge_{i \in S} \varphi_i \wedge \bigwedge_{i \notin S} \psi_i : S \subseteq \{1, \ldots, n\}\}$

The easiest proofs of these lemmas are semantic, using completeness. Of course there are also syntactic proofs.

2.1 Extending the notation to the modal setting

In this section, we expand our discussion to the case of formulas built in the basic modal similarity type. That is, we add a single modal operator $\Box$ to the syntax of propositional logic, generating formulas such as $\Box(p_{23} \wedge \neg \Box p_3)$. As always, we write $\Diamond \varphi$ for $\neg \Box \neg \varphi$. Let $\psi_1, \ldots, \psi_m$ be modal formulas. For each $S \subseteq \{\psi_1, \ldots, \psi_m\}$, let

$$\alpha_S \quad = \quad \bigwedge_{\psi_i \in S} \Diamond \psi_i \wedge \bigwedge_{\psi_i \notin S} \neg \Diamond \psi_i.$$

Also, let

$$\widehat{S} \quad = \quad \bigwedge_{\psi_i \in S} \Diamond \psi_i \wedge \Box \bigvee_{\psi_i \in S} \psi_i. \tag{2}$$

We remind the reader of the convention that $\bigvee \emptyset = \mathsf{F}$ and $\bigwedge \emptyset = \mathsf{T}$. We also remind the reader that K is the logical system extending propositional logic with K-axioms $\Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$ and with the rule of Necessitation: from φ, infer $\Box \varphi$. We write $\vdash \varphi$ for derivability in K.

Lemma 2.4. *Suppose that* $\vdash \oplus(\psi_1, \ldots, \psi_m)$. *Then in* K, $\vdash \oplus\{\alpha_S : S \subseteq \{\psi_1, \ldots, \psi_m\}\}$, *and also* $\vdash \oplus\{\widehat{S} : S \subseteq \{\psi_1, \ldots, \psi_m\}\}$.

Proof The first part is immediate from Lemma 2.3; the point is that $\vdash \oplus(\Diamond \psi_i, \neg \Diamond \psi_i)$ for all i. For the second, we show that $\vdash \alpha_S \leftrightarrow \widehat{S}$ for all S. Note that since $\vdash \oplus(\psi_1, \ldots, \psi_m)$, we also have $\vdash \bigvee\{\psi_1, \ldots, \psi_m\}$. By Necessitation, $\vdash \Box \bigvee\{\psi_1, \ldots, \psi_m\}$. So

$$\vdash \bigwedge_{\psi_i \notin S} \neg \Diamond \psi_i \to \Box \bigvee_{\psi_i \in S} \psi_i.$$

This implies that $\vdash \alpha_S \to \widehat{S}$. We prove also the converse. Since $\vdash \oplus(\psi_1, \ldots, \psi_m)$, we have

$$\vdash \bigvee_{\psi_i \in S} \psi_i \to \bigwedge_{\psi_i \notin S} \neg\psi_i.$$

So

$$\vdash \Box \bigvee_{\psi_i \in S} \psi_i \to \Box \bigwedge_{\psi_i \notin S} \neg\psi_i.$$

This easily leads to $\vdash \widehat{S} \to \alpha_S$. $\dashv$

2.2 Canonical formulas in modal logic

To generalize the notion of a state description to modal logic, we not only have to keep track of which atomic propositions are used in a given formulas, we also need to take note of the modal height.

Definition We define the *height* and *order* of an arbitrary formula φ of modal logic by the following recursions:

$$\begin{array}{lcl lcl}
ht(p_n) & = & 0 & ord(p_n) & = & n \\
ht(\mathsf{T}) & = & 0 & ord(\mathsf{T}) & = & 0 \\
ht(\mathsf{F}) & = & 0 & ord(\mathsf{F}) & = & 0 \\
ht(\neg\varphi) & = & ht(\varphi) & ord(\neg\varphi) & = & ord(\varphi) \\
ht(\varphi \wedge \psi) & = & max(ht(\varphi), ht(\psi)) & ord(\varphi \wedge \psi) & = & max(ord(\varphi), ord(\psi)) \\
ht(\Box\varphi) & = & 1 + ht(\varphi) & ord(\Box\varphi) & = & ord(\varphi)
\end{array}$$

The height (also called *depth*) measures the maximum nesting depth of boxes, and the order gives the largest subscript on any atomic proposition occurring. We also let

$$\mathcal{L}_{h,n} \quad = \quad \{\varphi : ht(\varphi) \leq h, ord(\varphi) \leq n\}.$$

For example,

$$\begin{array}{lcl}
ht(\Diamond p_3 \wedge \Box\Diamond p_2) & = & 2 \\
ord(\Diamond p_3 \wedge \Box\Diamond p_2) & = & 3
\end{array}$$

So that $\Diamond p_3 \wedge \Box\Diamond p_2$ belongs to $\mathcal{L}_{2,3}$. Indeed it belongs to $\mathcal{L}_{h,n}$ for $h \geq 2$ and $n \geq 3$.

Definition Fix a natural number m, and consider the first m atomic propositions $p_1, \ldots, p_n$. For each $T \subseteq \{p_1, \ldots, p_n\}$, let

$$\widehat{T} \quad = \quad \bigwedge_{p_i \in T} p_i \wedge \bigwedge_{p_i \notin T} \neg p_i \tag{3}$$

For example, with $n = 4$ and $T = \{p_1, p_4\}$, we have

$$\widehat{T} \quad = \quad p_1 \wedge p_4 \wedge \neg p_2 \wedge \neg p_3.$$

Note that $\mathsf{SD}_n = \{\widehat{T} : T \subseteq \{p_1, \ldots, p_n\}\}$.

Definition We define the sets $\mathcal{C}_{h,n}$ of *canonical formulas of height h and order n* as follows:

$\mathcal{C}_{0,n} = \mathsf{SD}_n$. Given $\mathcal{C}_{h,n}$, we let $\mathcal{C}_{h+1,n}$ be the collection of formulas of the form $\widehat{S} \wedge \widehat{T}$. The notation $\widehat{S}$ comes from (2), and the notation $\widehat{T}$ from (3). So for $S \subseteq \mathcal{C}_{h,n}$ and $T \subseteq \{p_1, \ldots, p_n\}$,

$$\begin{array}{rcl} \widehat{S} & = & (\bigwedge_{\psi \in S} \Diamond\psi) \wedge (\Box \bigvee_{\psi \in S} \psi) \\ \widehat{T} & = & (\bigwedge_{p_i \in T} p_i) \wedge (\bigwedge_{p_i \notin T} \neg p_i) \end{array}$$

Put differently, each $\alpha \in \mathcal{C}_{h+1,n}$ is of the form

$$(\bigwedge_{\psi \in S} \Diamond\psi) \ \wedge \ (\Box \bigvee S) \ \wedge \ (\bigwedge T) \ \wedge \ (\bigwedge_{p_i \notin T} \neg p_i) \tag{4}$$

for some $S \subseteq \mathcal{C}_{h,n}$ and some $T \subseteq \{p_1, \ldots, p_n\}$.

Proposition 2.5. *For each h and n, $\mathcal{C}_{h,n}$ is a finite subset of $\mathcal{L}_{h,n}$. Moreover, if $F(0,n) = 2^n$ and $F(h+1,n) = 2^{F(h,n)+n}$, then $|\mathcal{C}_{h,n}| = F(h,n)$.*

Example 2.1. $C_{0,1} = \{p_1, \neg p_1\}$. $\mathcal{C}_{1,1}$ is a set with eight elements. Because we refer to these elements at various points, it makes sense to adopt names for them. And because we are dealing with $n = 1$, we drop the subscript on p_1.

$$\begin{array}{rclcrcl} \alpha_1 & = & \widehat{\emptyset} \wedge p & \quad & \alpha_5 & = & \widehat{\{\neg p\}} \wedge p \\ \alpha_2 & = & \widehat{\emptyset} \wedge \neg p & & \alpha_6 & = & \widehat{\{\neg p\}} \wedge \neg p \\ \alpha_3 & = & \widehat{\{p\}} \wedge p & & \alpha_7 & = & \widehat{C_{0,1}} \wedge p \\ \alpha_4 & = & \widehat{\{p\}} \wedge \neg p & & \alpha_8 & = & \widehat{C_{0,1}} \wedge \neg p \end{array}$$

We have used the notation from above. For example, $\widehat{\{p\}}$ abbreviates $\Diamond p \wedge \Box p$, and $\widehat{\mathcal{C}_{0,1}}$ abbreviates $\Diamond p \wedge \Diamond\neg p \wedge \Box(p \vee \neg p)$. (The last conjunct is redundant, so it is better to think of $\widehat{\mathcal{C}_{0,1}}$ as $\Diamond p \wedge \Diamond \neg p$.)

Example 2.2. Let $\mathbb{A}$ be any Kripke model. Fix a number n. For every $a \in A$ and every h, we define the formula φ_a^h. The definition is by recursion on h (simultaneously for all $a \in A$) as follows: φ_a^0 is the unique canonical formula of height 0 and order n satisfied by a. (It is the conjunction of all atomic propositions satisfied by a and all negations of atomic propositions not satisfied by a.) Given φ_b^h for all $b \in A$, we define

$$\varphi_a^{h+1} \quad = \quad \bigwedge_{a \to b} \Diamond\varphi_b^h \ \wedge \ \Box \bigvee_{a \to b} \varphi_b^h \ \wedge \ \varphi_a^0.$$

Then each φ_a^h belongs to $\mathcal{C}_{h,n}$.

We shall see later that very canonical formula can be obtained in the manner of Example 2.2.

Lemma 2.6. *For all h and n, $\vdash \oplus\mathcal{C}_{h,n}$ in K. As a result, every world of every Kripke model satisfies a unique element of $\mathcal{C}_{h,n}$.*

Proof For $h = 0$, use Lemma 2.1. Assume that $\vdash \oplus\mathcal{C}_{h,n}$. By Lemma 2.4, $\vdash \oplus\{\widehat{S} : S \subseteq \mathsf{SD}_n\}$. Continuing, we have already seen that $\vdash \oplus\mathsf{SD}_n$. That is, $\vdash \oplus\{\widehat{T} : T \subseteq \{p_1, \ldots, p_n\}\}$. So by Lemma 2.2, we have

$$\vdash \oplus\{\widehat{S} \wedge \widehat{T} : S \subseteq \mathcal{C}_{h,n} \text{ and } T \subseteq \{p_1, \ldots, p_n\}\}.$$

That is, $\vdash \oplus\mathcal{C}_{h+1,n}$. $\dashv$

We next present the fact that justifies thinking of the canonical formulas as analogs of state descriptions. This result is the most important fact in this section, and it will be used without specific justification in the rest of this paper.

Lemma 2.7. *Let $\chi \in \mathcal{L}_{h,n}$ and $\alpha \in \mathcal{C}_{h,n}$. Then in K, either $\vdash \alpha \to \chi$ or else $\vdash \alpha \to \neg\chi$.*

Proof By induction on χ. All of the work is in the induction step for $\Box$. So we assume our lemma for χ and prove it for $\Box\chi$. Let h and n be large enough so that $\Box\chi \in \mathcal{L}_{h,n}$, and let $\alpha \in \mathcal{C}_{h,n}$. We must have $h > 0$. Write α as in (4). For each $\beta \in S$, the induction hypothesis applies to χ and β. We have two cases. First, assume that for some $\beta \in S$, $\vdash \beta \to \neg\chi$. Then $\vdash \Diamond\beta \to \Diamond\neg\chi$. The definition of α in (4) implies that $\vdash \alpha \to \Diamond\beta$. So $\vdash \alpha \to \Diamond\neg\chi$. We have $\vdash \alpha \to \neg\Box\chi$. The other case is when for each $\beta \in S$, $\vdash \beta \to \chi$. Then $\vdash \bigvee_{\beta\in S} \beta \to \chi$. So we also have $\vdash \Box \bigvee_{\beta\in S} \beta \to \Box\chi$. Again in view of (4), $\vdash \alpha \to \Box\chi$. This completes the induction step for $\Box$, and hence the overall proof. ⊣

With the foregoing definitions in place, we now take up the main thread of this paper, the construction of finite models from canonical formulas.

3 The models $\mathbb{B}_{h,n}$

Our first class of models is called $\mathbb{B}_{h,n}$. For natural numbers h and n, $\mathbb{B}_{h,n}$ might be called the *canonical finite model of height* $\leq h$ *and order* n. These models are *not* the central objects of study in this paper; those will be the models $\mathbb{C}_{h,n}(L)$ introduced in the next section. Our mention of the models $\mathbb{B}_{h,n}$ is mostly for completeness.

We define $\mathbb{B}_{h,n} = (B_{h,n}, \rightsquigarrow)$ as follows:

1. The worlds $B_{h,n}$ of $\mathbb{B}_{h,n}$ are the canonical formulas of height $\leq h$ and order n:

$$B_{h,n} \quad = \quad \mathcal{C}_{0,n} \cup \mathcal{C}_{1,n} \cup \cdots \cup \mathcal{C}_{h,n}.$$

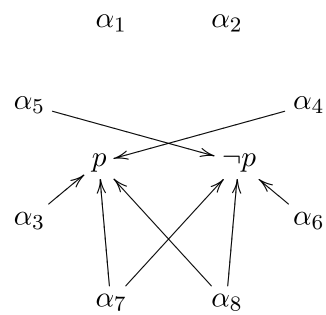
2. If α belongs to $\mathcal{C}_{i+1,n}$, say $\alpha = \widehat{S} \wedge \widehat{T}$, then for all $\beta \in S$, $\alpha \rightsquigarrow \beta$. These are the only $\rightsquigarrow$ relations in the model.

3. For $1 \leq j \leq n$, $v(p_j)$ is all of the canonical formulas of height $\leq h$ and order n in which p_j is a conjunct (rather than $\neg p_j$). An equivalent formulation is

$$v(p_j) \quad = \quad \{\alpha \in \mathcal{C}_{\leq h,n} \ : \vdash \alpha \to p_j\}.$$

The reason why we use the symbol $\rightsquigarrow$ for the accessibility relation in a model rather than just $\to$ is to avoid the confusion of logical implication with the accessibility relation in our formula-based models.

Example 3.1. Here is $\mathbb{B}_{1,1}$, using the notation from Example 2.1.



The atomic proposition p is true at the points α_1, α_3, α_5, α_7, and p. We have pictured $\mathcal{C}_{0,1}$ in the middle, and $\mathcal{C}_{1,1}$ outside. Incidentally, there is no significance to the arrangement of $\alpha_1, \ldots, \alpha_8$ in the picture. We maintain this arrangement in several figures in the sequel, and we do so mainly because it results in pictures of partial orders that grow from the bottom upwards.

Lemma 3.1. *For all* $\alpha \in \mathcal{C}_{\leq h,n}$, $(\mathbb{B}_{h,n}, \alpha) \models \alpha$.

Proof For $h = 0$, note that for $1 \leq i \leq n$, $\alpha \models p_i$ iff $p_i \in v(\alpha)$ iff $\vdash \alpha \to p_i$. Since α is a state description, $\vdash \alpha \to p_i$ or $\vdash \alpha \to \neg p_i$; and the choice depends only on whether p_i is or is not a conjunct of α. This easily leads to our result.

Assume the result for h, and let $\alpha \in \mathcal{C}_{h+1,n}$. Then α is of the form $\widehat{S} \wedge \widehat{T}$, with $S \subseteq \mathcal{C}_{h,n}$ and $T \subseteq \{p_1, \ldots, p_n\}$. The same argument as above shows that $\alpha \models \widehat{T}$. If $\beta \in S$, then $\alpha \rightsquigarrow \beta$. By induction hypothesis, $\beta \models \beta$, so $\alpha \models \Diamond\beta$. If $\beta \notin S$, then we claim that no child γ of α can satisfy β. For such γ would be different from β, and as $\gamma \models \gamma$ and every world satisfies a *unique* element of $\mathcal{C}_{h,n}$, we see that γ cannot satisfy β. Since this is true for all γ, we see that $\alpha \models \neg\Diamond\beta$. Thus, for all $\beta \in S$, $\alpha \models \Diamond\beta$, and for all $\beta \notin S$, $\alpha \models \neg\Diamond\beta$. Overall, $\alpha \models \widehat{S}$. $\dashv$

This last results explains our remark at the end of Example 2.2: every $\alpha \in \mathcal{C}_{h,n}$ is of the form φ_a^h for some pointed model $(\mathbb{A}, a)$.

Lemma 3.2. *Fix a number* n. *For every* $\chi \in \mathcal{L}_{h,n}$ *and every* $\alpha \in \mathcal{C}_{\leq h,n}$, *the following are equivalent:*

1. $(\mathbb{B}_{h,n}, \alpha) \models \chi$.
2. $\vdash \alpha \to \chi$ *in* K.

The proof is by induction on h.

Completeness and decidability of K We are not so interested in these models $\mathbb{B}_{h,n}$ in this paper, mostly because it is not possible to adapt them to give results about specific logics of interest. (The exceptions are for KB and KD, but we shall not go into details on those.) But we do want to mention that those models give the completeness and decidability of K, as noted by Fine [4].

Theorem 3.3 (Completeness). *If $\models \psi$, then $\vdash \psi$ in K.*

Proof Let $h = ht(\psi)$, and $n = ord(\psi)$. We assume that that $\models \psi$; i.e., ψ holds at all worlds of all models. In particular, for all $\alpha \in \mathcal{C}_{h,n}$, we have $(\mathbb{B}_{h,n}, \alpha) \models \psi$. We work in K. By Lemma 3.2, $\vdash \alpha \to \psi$ for all $\alpha \in \mathcal{C}_{h,n}$. But then by propositional logic, $\vdash \bigvee \mathcal{C}_{h,n} \to \psi$. And as we know from Lemma 2.6, $\vdash \bigvee \mathcal{C}_{h,n}$. So as desired, $\vdash \psi$. $\dashv$

Corollary 3.4 (Decidability). *It is decidable whether $\models \psi$ or not.*

Proof Let h be the height of ψ, and n the order. Consider $\mathbb{B}_{h,n}$. This model can be constructed (computably) from the numbers h and n. If we want to see whether ψ is true in *all* models, we need only look to see if it is true at all points in that one (big, but finite) model. And the evaluation is again a computable matter. $\dashv$

4 The models $\mathbb{C}_{h,n}(L)$

At this point we have completeness for K, and we would also like to have it for logics like T, $S4$, etc. For this, the model of the previous section is not good enough. Let L be a normal modal logic. When we have two logics, say L and M, we write $L \leq M$ to mean that every axiom of L is provable in M. We define $\mathbb{C}_{h,n}(L) = (C_{h,n}(L), \xrightarrow{c})$, the *canonical model of L-consistent fomulas of height h and order n*:

1. $C_{h,n}(L) = \{\alpha \in \mathcal{C}_{h,n} : \alpha \text{ is consistent in } L\}$.
2. $\alpha \xrightarrow{c} \beta$ iff $\alpha \wedge \Diamond\beta$ is consistent in L.
3. $v(p_i) = \{\alpha : \vdash \alpha \to p_i\}$.

The important thing to notice first is that the $\xrightarrow{c}$ relation here is quite different from the relation $\rightsquigarrow$ in $\mathbb{B}_{h,n}$. Our relation $\xrightarrow{c}$ comes from Kozen and Parikh [7]; it is the central way in which our development of the subject differs from that of Fine [4]. Again, the symbol for the accessibility relation was chosen to avoid the confusion with logical implication. (And we don't use $\rightsquigarrow$ to avoid confusion with the models $\mathbb{B}_{h,n}$.) The particular relation $\xrightarrow{c}$ has an intuition behind it. We take the canonical sentences to be "maximally informative" (up to their height and order). The main quest in this paper is for a model construction using them. Now in any Kripke model of L, say $\mathbb{A} = (A, \to)$, if $a \to b$, then $\alpha_a \wedge \Diamond\alpha_b$ is satisfiable and hence consistent in L, where α_a and α_b are the sentences of $\mathcal{C}_{h,n}$ satisfied by a and b in $\mathbb{A}$, respectively. Hence $\alpha_a \xrightarrow{c} \alpha_b$. But we don't want to define $\xrightarrow{c}$ in terms of models; we prefer a definition that is more "syntactic". And so instead of saying $\alpha \xrightarrow{c} \beta$ iff there is some L-model $\mathbb{A}$ containing $a \to b$ such that $\alpha_a = \alpha$ and $\alpha_b = \beta$, we use the consequence of this, that $\alpha \wedge \Diamond\beta$ be consistent in L.

4.1 Examples

This section constructs $\mathbb{C}_{h,n}(L)$ for various values of h and L. (In all of our examples, $n = 1$. So instead of writing p_1 or $\neg p_1$, we simply write p or $\neg p$.) The case of $h = 0$ is degenerate in our examples: $\mathcal{C}_{0,n}$ is a complete model on 2^n points. However, this is neither interesting nor important: our results would do not use the structure of $\mathcal{C}_{0,n}$, and so we could re-define things and still obtain the same overall results. Most of our examples are shown in Figure 1.

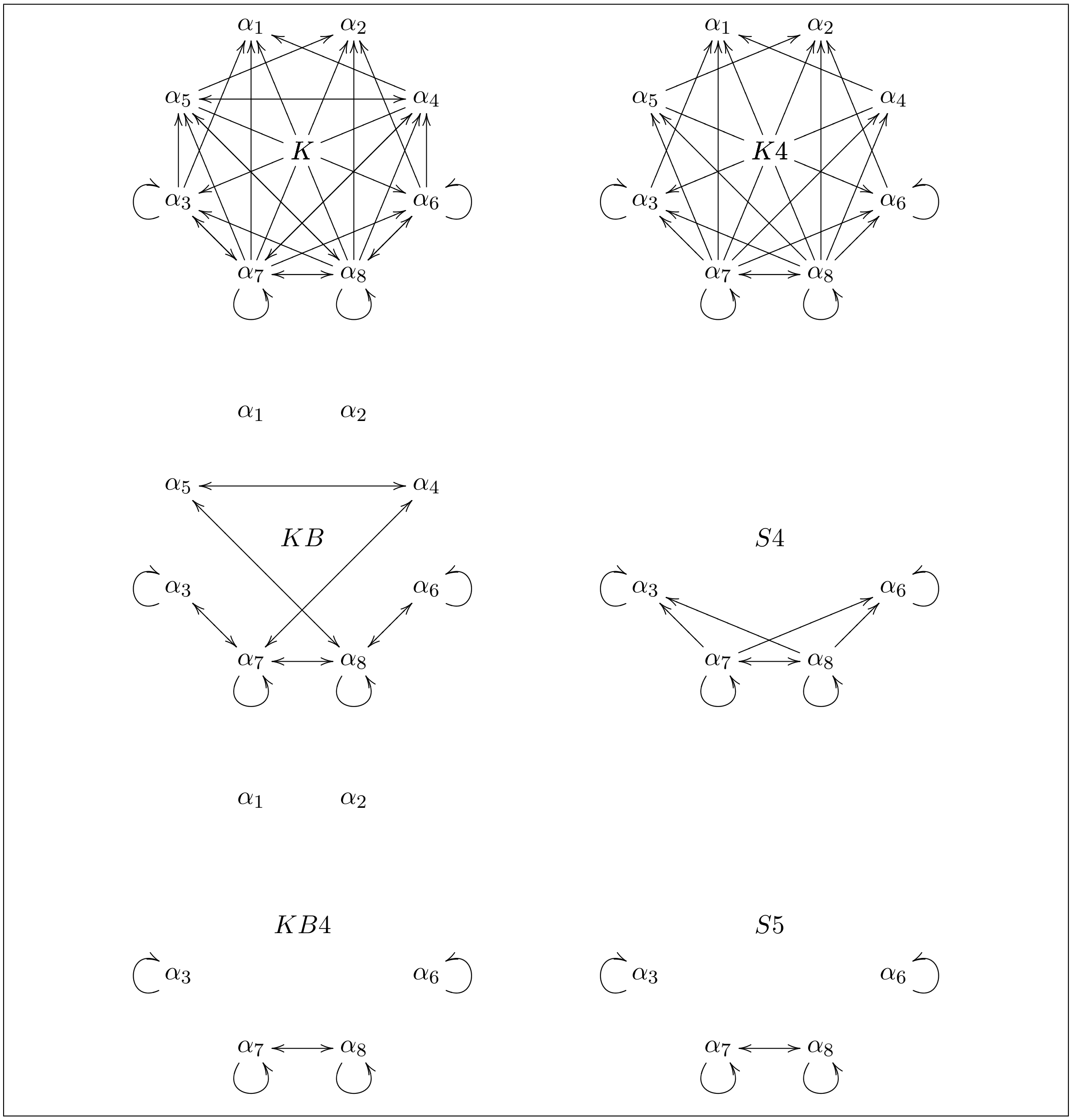


Figure 1: $\mathbb{C}_{1,1}(L)$ for various logics L. The formulas $\alpha_1, \ldots, \alpha_8$ are from Example 2.1.

First we construct the model $\mathbb{C}_{1,1}(K)$. The eight elements $\mathcal{C}_{1,1}$ are all consistent, and we listed them with names in Example 2.1. The points satisfying p are those on the left side of the figure: α_1, α_3, α_5, and α_7. We retain this convention in all of the models pictured in this paper.

Here are a few comments that go with Figure 1. All eight formulas $\alpha_1, \ldots, \alpha_8$ are all consistent in $K4$. But we lose some arrows in $\mathcal{C}_{1,1}(K4)$. Specifically, we lose $\alpha_5 \to \alpha_4$, $\alpha_5 \to \alpha_8$, $\alpha_4 \to \alpha_5$, $\alpha_4 \to \alpha_7$, $\alpha_3 \to \alpha_7$, and $\alpha_6 \to \alpha_8$.

Of the eight elements of $\mathcal{C}_{1,1}$, only four are consistent in $S4$. $\mathbb{C}_{1,1}(KT)$ turns out to be the same as $\mathbb{C}_{1,1}(S4)$. In $\mathbb{C}_{1,1}(KB)$ and $\mathbb{C}_{1,1}(KB4)$, the points α_1 and α_2 have no outgoing arrows.

Example 4.1. Using our work on $\mathbb{C}_{1,1}(S4)$, we construct $\mathbb{C}_{2,1}(S4)$. We begin with the following elements:

$$
\begin{array}{lll@{\qquad}lll}
\beta_1 & = & \{\alpha_3, \alpha_6, \alpha_7, \alpha_8\}\widehat{} \wedge p & \beta_8 & = & \{\alpha_3, \alpha_6, \alpha_7, \alpha_8\}\widehat{} \wedge \neg p \\
\beta_2 & = & \{\alpha_3, \alpha_6, \alpha_7\}\widehat{} \wedge p & \beta_9 & = & \{\alpha_3, \alpha_6, \alpha_8\}\widehat{} \wedge \neg p \\
\beta_3 & = & \{\alpha_3, \alpha_7, \alpha_8\}\widehat{} \wedge p & \beta_{10} & = & \{\alpha_3, \alpha_7, \alpha_8\}\widehat{} \wedge \neg p \\
\beta_4 & = & \{\alpha_6, \alpha_7, \alpha_8\}\widehat{} \wedge p & \beta_{11} & = & \{\alpha_6, \alpha_7, \alpha_8\}\widehat{} \wedge \neg p \\
\beta_5 & = & \{\alpha_6, \alpha_7\}\widehat{} \wedge p & \beta_{12} & = & \{\alpha_3, \alpha_8\}\widehat{} \wedge \neg p \\
\beta_6 & = & \{\alpha_7, \alpha_8\}\widehat{} \wedge p & \beta_{13} & = & \{\alpha_7, \alpha_8\}\widehat{} \wedge \neg p \\
\beta_7 & = & \{\alpha_3\}\widehat{} \wedge p & \beta_{14} & = & \{\alpha_6\}\widehat{} \wedge \neg p
\end{array}
$$

These are the elements of $\mathcal{C}_{2,1}$ consistent in $S4$. The structure as always is given by $\beta_i \xrightarrow{c} \beta_j$ iff $\beta_i \wedge \Diamond\beta_j$ is consistent in $S4$.

It does take a lot of work to get the full structure, and we admit that we got the help of a computer program for this.[1] A picture of the model is shown in Figure 2. The picture does not show the reflexive arrows on all points, or the arrows from β_1 and β_2 to all points. It also does not include the information that p is true exactly at β_1, …, β_7.

4.2 Properties of $\mathbb{C}_{h,n}(L)$ and easy completeness results

The models $\mathbb{C}_{h,n}(L)$ are our main objects of study.

Lemma 4.1. *In L, $\vdash \oplus\mathcal{C}_{h,n}(L)$.*

Proof As we know from Lemma 2.6, in K, $\vdash \oplus\mathcal{C}_{h,n}$. And working in L, if α is not consistent then $\vdash \neg\alpha$. $\dashv$

Lemma 4.2. *Let $\psi \in \mathcal{L}_{h,n}$. Then*

1. *In K, $\vdash \psi \leftrightarrow \bigvee\{\alpha \in \mathcal{C}_{h,n} :$ in $K, \vdash \alpha \to \psi\}$.*
2. *In L, $\vdash \psi \leftrightarrow \bigvee\{\alpha \in \mathcal{C}_{h,n}(L) :$ in $K, \vdash \alpha \to \psi\}$.*

Proof Let S be the set of $\alpha \in \mathcal{C}_{h,n}$ such that $\vdash \alpha \to \psi$. Then $\vdash \bigvee S \to \psi$. And for $\alpha \notin S$, $\vdash \alpha \to \neg\psi$. So $\vdash \psi \leftrightarrow \bigvee_{\alpha\in\mathcal{C}_{h,n}}(\alpha \wedge \psi) \leftrightarrow \bigvee_{\alpha\in S}(\alpha \wedge \psi) \leftrightarrow \bigvee_{\alpha\in S} \alpha$.

(2) follows from (1). and Lemma 4.1. $\dashv$

[1]We used the Logics Workbench: `http://www.lwb.unibe.ch/`

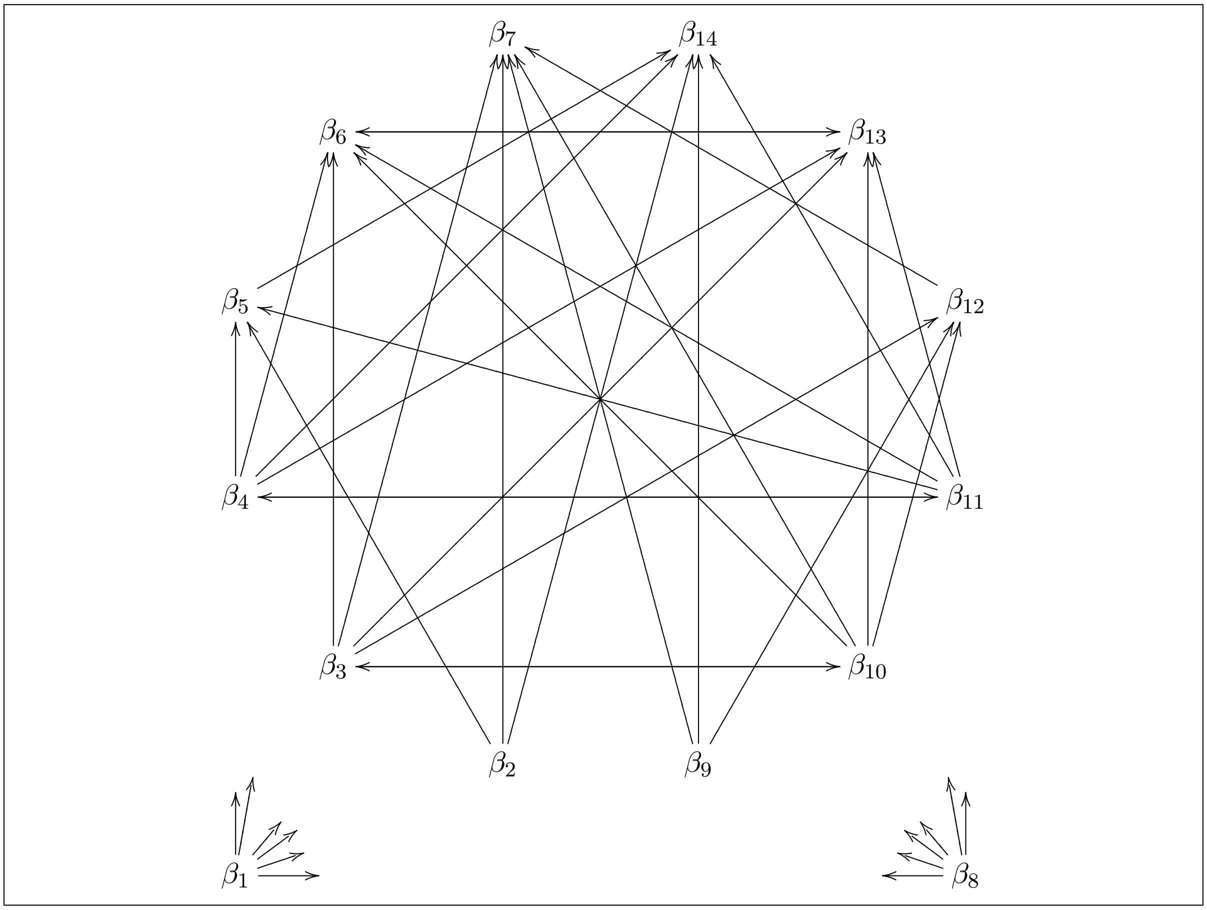


Figure 2: $\mathbb{C}_{2,1}(S4)$, omitting all self-loops and most arrows from β_1 and β_8.

Lemma 4.3 (Existence Lemma)**.** *Let $\psi \in \mathcal{L}_{h,n}$, let φ be arbitrary, and suppose that $\varphi \wedge \Diamond\psi$ is consistent in L. Then there is some $\alpha \in \mathcal{C}_{h,n}(L)$ such that $\varphi \wedge \Diamond\alpha$ is consistent in L, and $\vdash \alpha \to \psi$ in K.*

Proof By Lemma 4.2, we see that in L,

$$\vdash \Diamond\psi \leftrightarrow \bigvee\{\Diamond\alpha : \alpha \in \mathcal{C}_{h,n}(L) \text{ and in } K, \vdash \alpha \to \psi\}.$$

So

$$\vdash \varphi \wedge \Diamond\psi \leftrightarrow \bigvee\{\varphi \wedge \Diamond\alpha : \alpha \in \mathcal{C}_{h,n}(L) \text{ and in } K, \vdash \alpha \to \psi\}.$$

By our assumption, the finite disjunction on the right just above is consistent in L. So at least one disjunct must be consistent. This gives some $\alpha \in \mathcal{C}_{h,n}(L)$ with the desired properties. $\dashv$

Lemma 4.4 (Truth Lemma for $\mathbb{C}_{h,n}(L)$)**.** *For all $\alpha \in \mathcal{C}_{h,n}(L)$ and all $\psi \in \mathcal{L}_{h,n}$, $(\mathbb{C}_{h,n}(L), \alpha) \models \psi$ iff $\vdash \alpha \to \psi$ in K.*

Proof By induction on ψ. The only interesting step is for $\Diamond\psi$, assuming the result for ψ. Let h and n be such that $\Diamond\psi \in \mathcal{L}_{h,n}$. Note that $\psi \in \mathcal{L}_{h,n}$ as well.

Assume that $(\mathbb{C}_{h,n}(L), \alpha) \models \Diamond\psi$. Let $\beta \in \mathcal{C}_{h,n}(L)$ be such that $\alpha \wedge \Diamond\beta$ is consistent in L and $(\mathbb{C}_{h,n}(L), \beta) \models \psi$. By induction hypothesis, $\vdash \beta \to \psi$. Thus $\vdash \Diamond\beta \to \Diamond\psi$. We thus see that $\alpha \wedge \Diamond\psi$ is consistent in L. As we know, either $\vdash \alpha \to \Diamond\psi$ (in K), or $\vdash \alpha \to \neg\Diamond\psi$ (in K). But the latter case cannot happen, because it would contradict the consistency of $\alpha \wedge \Diamond\psi$.

In the other direction, suppose that $\vdash \alpha \to \Diamond\psi$. Since α is consistent in L, so is $\alpha \wedge \Diamond\psi$. So by Existence Lemma 4.3, there is some $\beta \in \mathcal{C}_{h,n}(L)$ such that $\alpha \wedge \Diamond\beta$ is consistent in L, and $\vdash \beta \to \psi$ in K. Thus $\alpha \xrightarrow{c} \beta$ in $\mathbb{C}_{h,n}(L)$, and by induction hypothesis, $(\mathbb{C}_{h,n}(L), \beta) \models \psi$. So $(\mathbb{C}_{h,n}(L), \alpha) \models \Diamond\psi$. $\dashv$

As a result of Lemma 4.4, $(\mathbb{C}_{h,n}(L), \alpha) \models \alpha$ for all $\alpha \in \mathcal{C}_{h,n}(L)$.

Lemma 4.5. *The following hold for all h and n:*

1. *If $KT \leq L$, $\mathbb{C}_{h,n}(L)$ is reflexive.*
2. *If $KD \leq L$, $\mathbb{C}_{h,n}(L)$ is serial.*
3. *If $KB \leq L$, $\mathbb{C}_{h,n}(L)$ is symmetric.*

Proof For (1), let $\alpha \in \mathcal{C}_{h,n}(L)$. We must show that $\alpha \xrightarrow{c} \alpha$; i.e., $\alpha \wedge \Diamond\alpha$ is consistent. If not, then $\vdash \alpha \to \Box\neg\alpha$. But by T, $\vdash \Box\neg\alpha \to \neg\alpha$. So $\vdash \alpha \to \neg\alpha$. This contradicts the consistency of α in the logic L.

Next, we prove (2). Let $\alpha \in \mathcal{C}_{h,n}(L)$, so $\alpha \wedge \Diamond\top$ is consistent in KD. By the Existence Lemma 4.3, there is some $\beta \in \mathcal{C}_{h,n}(L)$ such that $\alpha \wedge \Diamond\beta$ is consistent. Thus $\alpha \xrightarrow{c} \beta$ in $\mathbb{C}_{h,n}(L)$.

Finally, we show that $\mathcal{C}_{h,n}(L)$ is symmetric whenever $KB \leq L$. Suppose that $\alpha \wedge \Diamond\beta$ is consistent, but $\vdash \beta \to \Box\neg\alpha$. Then $\vdash \Diamond\beta \to \Diamond\Box\neg\alpha$. By B, $\vdash \alpha \to \Box\Diamond\alpha$. So $\Box\Diamond\alpha \wedge \Diamond\Box\neg\alpha$ is consistent (in L). This is clearly a contradiction. $\dashv$

There is one final general property of the models $\mathcal{C}_{h,n}$.

Lemma 4.6. $\mathbb{C}_{h,n}(L)$ *is strongly extensional: every bisimulation on it is a subrelation of the identity.*

Proof Let R be a bisimulation on $\mathbb{C}_{h,n}(L)$, and suppose that $\alpha \; R \; \beta$. Then α and β satisfy the same modal formulas. By Lemma 4.4, $\alpha \models \alpha$. So $\beta \models \alpha$. Then by Lemma 4.4 again, we have that in K, $\vdash \beta \to \alpha$. In view of Lemma 4.1 above, we must have $\beta = \alpha$; that is, distinct canonical formulas imply each other's negation. $\dashv$

The results of this section give completeness/decidability results for the logics KT, KD, KB, KTB, and KDB on the appropriate classes of structures. The proofs are routine modifications of Theorem 3.3 and Corollary 3.4. We illustrate this point with KB.

Theorem 4.7 (Completeness)**.** *If ψ holds at every world in every symmetric model, then $\vdash \psi$ in KB. Moreover, the property of being provable in KB is decidable.*

Proof Let h and n be large enough so that $\psi \in \mathcal{L}_{h,n}$. The model $\mathbb{C}_{h,n}(KB)$ is symmetric, and so each $\alpha \in \mathcal{C}_{h,n}(KB)$ satisfies ψ in it. So in K, $\vdash \bigvee \mathcal{C}_{h,n}(KB) \to \psi$. But in KB, $\vdash \bigvee \mathcal{C}_{h,n}(KB)$. Hence in KB, $\vdash \psi$.

The decidability is again a corollary to the proof above: ψ is provable in KB iff it holds at each world of $\mathbb{C}_{h,n}(KB)$, where the numbers h and n are obtained recursively from ψ. Now at this point we do not know that the model $\mathbb{C}_{h,n}(KB)$ itself is recursive in h and n. But given h and n, we can compute $m = F(h, n)$ as in Proposition 2.5. Then from this we compute all symmetric Kripke models on at most m nodes. From these we can tell which elements of $\mathcal{C}_{h,n}$ are consistent in KB. And since one of the symmetric models will indeed be $\mathbb{C}_{h,n}(KB)$, we can see for $\alpha, \beta \in \mathcal{C}_{h,n}(KB)$ whether or not $\alpha \wedge \Diamond\beta$ is satisfiable on a symmetric model of size $\leq m$ or not. This latter question is the same as whether of not $\alpha \wedge \Diamond\beta$ is consistent in KB, by completeness. So from m we can compute the structure of $\mathbb{C}_{h,n}(KB)$ by examining all possible symmetric models of size $\leq m$. $\dashv$

Here is a more abstract formulation of the last result.

Theorem 4.8. *Let L be a logic, let C be a class of finite structures, and assume the following conditions:*

1. *L is sound for C.*
2. *Membership in C is decidable.*
3. *Each $\mathcal{C}_{h,n}$ belongs to C.*

Then L is complete for C, and provability in L is decidable.

We emphasize that the decidability part of the last result is particularly crude.

5 Applications

This section contains applications of the models $\mathbb{C}_{h,n}(L)$ to the weak completeness of a number of modal propositional systems.

5.1 Logics built from 4, D, T, and B

The completeness results for $K4$ and the analog of Lemma 4.5 for the standard logics extending it are harder to come by. It is instructive to look at the argument for Lemma 4.5, and then to attempt to prove the same result for transitivity and $K4$. Let us assume that $\alpha \wedge \Diamond\beta$ and $\beta \wedge \Diamond\gamma$ are both consistent in $K4$; we must show that $\alpha \wedge \Diamond\gamma$ is also consistent in $K4$. As it happens, this *is* the case, but the argument is not the kind of "soft" argument we saw in Lemma 4.5. More to the point, the argument for KB in Lemma 4.5 did not use the fact that the formulas α and β belonged to $\mathcal{C}_{h,n}(KB)$: the fact there was a general fact that held for all formulas whatsoever. But for the harder results here, we actually need to use the fact that we are dealing with *canonical* formulas. This is because it is not in general true for any formulas φ, ψ, and χ, that the consistency in $K4$ of $\varphi \wedge \Diamond\psi$ and $\psi \wedge \Diamond\chi$ implies the consistency of $\varphi \wedge \Diamond\chi$. (Consider, for example, $\varphi = p \wedge \Box\Diamond p$, $\psi = \neg p$, and $\chi = \Box\neg p$.)

Despite this, one can show that $\mathbb{C}_{h,n}(K4)$ *is* transitive; see results later in this section. But the argument is indirect and goes through a model $\mathbb{M}_{h,n}(K4)$ which is easily seen to be transitive and later by an argument turns out to be the same as $\mathbb{C}_{h,n}(K4)$. The same strategy works for other logics, but different logics require different work. So we have an overall open question: is it true that for every logic L such that $K4 \leq L$, $\mathbb{C}_{h,n}(L)$ is transitive for almost all h? We believe the answer should be no, but we not aware of a counterexample.

In this section $\vdash \alpha$ means that α is provable in $K4$ (unless noted otherwise, say as provability in K).

Lemma 5.1. *For every $\alpha \in \mathcal{C}_{h+1,n}$ there is a unique $\alpha' \in \mathcal{C}_{h,n}$, the* derivative of α, *such that in K, $\vdash \alpha \to \alpha'$. Moreover for all logics L, if $\alpha \in \mathcal{C}_{h+1,n}(L)$, then $\alpha' \in \mathcal{C}_{h,n}(L)$.*

Proof By Lemma 2.6, there is some $\beta \in \mathcal{C}_{h,n}$ such that $(\mathbb{C}_{h+1,n}, \alpha) \models \beta$. But β must be unique, since we also have $\models \oplus\mathcal{C}_{h,n}$. $\dashv$

Example 5.1. We use the notation from Examples 2.1 and 4.1. First, $\alpha_1' = \alpha_3' = \alpha_5' = \alpha_7' = p$, and $\alpha_2' = \alpha_4' = \alpha_6' = \alpha_8' = \neg p$. Using this, we have $\beta_1' = \beta_2' = \cdots = \beta_6' = \alpha_7$, $\beta_7' = \alpha_3$, $\beta_8' = \cdots = \beta_{13}' = \alpha_8$, and $\beta_{14}' = \alpha_6$.

Lemma 5.2. *Let $\alpha \xrightarrow{c} \beta$ in $\mathbb{C}_{h+1,n}(L)$. Then in K, $\vdash \alpha \to \Diamond\beta'$.*

Proof Assume that $\alpha \wedge \Diamond\beta$ is consistent. Since $\vdash \beta \to \beta'$, we see that $\alpha \wedge \Diamond\beta'$ is also consistent. Note that $\Diamond\beta' \in \mathcal{L}_{h+1,n}$. Therefore $\vdash \alpha \to \Diamond\beta'$. $\dashv$

Lemma 5.3. *Let $K4 \leq L$ and let $\alpha \xrightarrow{c} \beta$ in $\mathbb{C}_{h+1,n}(L)$. Then if $\varphi \in \mathcal{L}_{h,n}$ and $\vdash \beta \to \Diamond\varphi$, then $\vdash \alpha \to \Diamond\varphi$.*

Proof Suppose that $\vdash \beta \to \Diamond\varphi$. Then again $\alpha \wedge \Diamond\Diamond\varphi$ is consistent. But in $K4$, $\vdash \Diamond\Diamond\varphi \to \Diamond\varphi$, Therefore $\alpha \wedge \Diamond\varphi$ is consistent, so $\vdash \alpha \to \Diamond\varphi$. $\dashv$

As mentioned above, our study of the models $\mathbb{C}_{h,n}(L)$ goes via the study of models which are prima facie different from $\mathbb{C}_{h,n}(L)$. The main ones are the models $\mathbb{M}_{h,n}(L) = (\mathcal{C}_{h,n}, \xrightarrow{m})$, defined as follows:

1. $\mathbb{M}_{0,n}(L) = \mathbb{C}_{0,n}(L)$.

2. The valuation on $\mathbb{M}_{h+1,n}(L)$ again has $v(p_i) = \{\alpha\ : \vdash \alpha \to p_i\}$

3. The accessibility relation on $\mathbb{M}_{h+1,n}(L)$ is defined by $\alpha \overset{m}{\to} \beta$ iff

   (a) $\vdash \alpha \to \Diamond\beta'$, and

   (b) whenever $\gamma \in \mathcal{C}_{h,n}$ is such that $\vdash \beta \to \Diamond\gamma$, then also $\vdash \alpha \to \Diamond\gamma$.

Incidentally, as with all our models, the case of $h = 0$ is not so significant. We could have defined the structure differently for $h = 0$ and very little would have changed.

Lemma 5.4. *Let $K4 \le L$. If $\alpha \overset{c}{\to} \beta$, then $\alpha \overset{m}{\to} \beta$.*

Proof This follows from Lemmas 5.2 and 5.3. ⊣

Lemma 5.5. *Concerning the models $\mathbb{M}_{h,n}(L)$:*

1. *$\mathbb{M}_{h,n}(L)$ is transitive.*

2. *If $KT \le L$, then $\mathbb{M}_{h,n}(L)$ is reflexive.*

3. *If $KD \le L$, then $\mathbb{M}_{h,n}(L)$ is serial.*

Proof $\mathbb{M}_{0,n}(L)$ is easily transitive. We therefore consider only the models $\mathbb{M}_{h+1,n}(L)$. Suppose that $\alpha \overset{m}{\to} \beta \overset{m}{\to} \gamma$. We show that $\alpha \overset{m}{\to} \gamma$. Since $\beta \overset{m}{\to} \gamma$, we have $\vdash \beta \to \Diamond\gamma'$. It follows from this and the fact that $\alpha \overset{m}{\to} \beta$ that $\vdash \alpha \to \Diamond\gamma'$. This is half of what we want. For the other half, suppose that $\vdash \gamma \to \Diamond\delta$. Then $\vdash \beta \to \Diamond\delta$, and so also $\vdash \alpha \to \Diamond\delta$.

Turning to the second part, $KT \le L$, then $\vdash \alpha \to \Diamond\alpha'$ for all $\alpha \in \mathcal{C}_{h,n}(L)$. Consequently, $M_{h,n}(L)$ is reflexive.

For part (3), suppose that $KD \le L$. Let $\alpha \in \mathcal{C}_{h,n}(L)$. Then $\vdash \alpha \to \Diamond\mathsf{T}$. By the Existence Lemma, let $\beta \in \mathcal{C}_{h,n}(L)$ be such that $\alpha \overset{c}{\to} \beta$. Then also $\alpha \overset{m}{\to} \beta$. ⊣

Lemma 5.6. *Each model $\mathbb{M}_{h,n}(L)$ is strongly extensional.*

We shall need Truth Lemma for the models $\mathbb{M}_{h,n}(L)$. But it makes sense to prove a a generalized version of the Truth Lemma, based on the following definition.

Definition A relation $\rightsquigarrow$ on $\mathcal{C}_{h,n}(L)$ is *suitable* if the following two properties hold:

1. If $\alpha \overset{c}{\to} \beta$ in $\mathcal{C}_{h,n}(L)$, then $\alpha \rightsquigarrow \beta$.

2. If $\alpha \rightsquigarrow \beta$, then $\vdash \alpha \to \Diamond\beta'$.

The main example so far of a suitable relation is $\overset{m}{\to}$ on $\mathcal{C}_{h,n}(K4)$. But there are others, for example the symmetric closure of $\overset{m}{\to}$ on $\mathcal{C}_{h,n}(KB4)$. For the connection of this notion to filtration, see Section 7.1.

Lemma 5.7 (A Generalized Truth Lemma)**.** *Let $\rightsquigarrow$ be a suitable relation on $\mathbb{C}_{h,n}(L)$. Then for all $\alpha \in \mathcal{C}_{h,n}(L)$ and all $\psi \in \mathcal{L}_{h,n}$, $((\mathcal{C}_{h,n}, \rightsquigarrow), \alpha) \models \psi$ iff $\vdash \alpha \to \psi$ in K.*

Proof By induction on ψ. ⊣

Lemma 5.8. $\mathbb{C}_{h,n}(K4) = \mathbb{M}_{h,n}(K4)$.

Proof We need only check that if $\alpha \xrightarrow{m} \beta$, then $\alpha \xrightarrow{c} \beta$. We apply Lemma 5.7 to $\mathbb{M}_{h,n}$. We see that $(\mathbb{M}_{h,n}(K4), \alpha) \models \alpha \wedge \Diamond\beta$. Since the model $\mathbb{M}_{h,n}(K4)$ is transitive, $\alpha \wedge \Diamond\beta$ is consistent in $K4$. Thus $\alpha \xrightarrow{c} \beta$. $\dashv$

Theorem 5.9. *Every $\alpha \in \mathcal{C}_{h,n}(K4)$ is satisfied on some finite transitive model. Thus $K4$ is complete for transitive models, and also decidable.*

Completeness of $S4$, $KD4$, $KB4$, and $S5$ At this point we extend our results to the standard logics which include the 4 axiom.

Lemma 5.10. $\mathbb{M}_{h,n}(S4) = \mathbb{C}_{h,n}(S4)$.

Proof As in Lemma 5.8, we observe that if $\alpha \xrightarrow{m} \beta$, then $\alpha \wedge \Diamond\beta$ is satisfied in the world α in the reflexive and transitive model $\mathbb{M}_{h,n}(S4)$ (see Lemma 5.5). Hence $\alpha \wedge \Diamond\beta$ is consistent in $S4$; i.e., $\alpha \xrightarrow{c} \beta$. $\dashv$

As a result of Lemma 5.10, $S4$ is complete for reflexive transitive models, and it is also decidable. A similar result holds for $KD4$. However, things are different for $KB4$:

Proposition 5.11. $\mathbb{M}_{1,1}(KB4) \neq \mathbb{C}_{1,1}(KB4)$.

Proof Let $\alpha = p \wedge \Diamond p \wedge \Diamond\neg p \wedge \Box(p \vee \neg p)$, and $\beta = \neg p \wedge \Box\mathsf{F}$. Both are consistent in $KB4$, since both are satisfiable in symmetric transitive models. Then $\alpha' = p$ and $\beta' = \neg p$. So $\vdash \alpha \to \Diamond\beta'$. In addition, there no γ such that $\vdash \beta \to \Diamond\gamma$. This shows $\alpha \xrightarrow{m} \beta$. But $\vdash (\alpha \wedge \Diamond\beta) \to \Diamond\Box\mathsf{F}$, and so in $KB4$, $\alpha \wedge \Diamond\beta$ is inconsistent.

A related point: $\xrightarrow{m}$ on $\mathbb{M}_{h,n}(KB4)$ is not symmetric: as we have seen, $\alpha \xrightarrow{m} \beta$. But $\not\vdash \beta \to \Diamond\alpha'$, so the converse does not hold. $\dashv$

To modify the work of this section to get the completeness result for $KB4$, we must therefore do more. Define a model $\mathbb{M}^{\leftrightarrow}_{h,n}(L)$ to be the same as $\mathbb{M}_{h,n}(L)$, except that

$$\alpha \xrightarrow{mm} \beta \quad \text{iff} \quad \text{both } \alpha \xrightarrow{m} \beta \text{ and } \beta \xrightarrow{m} \alpha.$$

From Lemma 5.4 we see that if $KB4 \leq L$ and $\alpha \xrightarrow{c} \beta$, then $\alpha \xrightarrow{mm} \beta$. The definition of $\xrightarrow{mm}$ implies that this relation is symmetric and transitive. Moreover, if $KT \leq L$, then $\xrightarrow{m}$ is also reflexive (see Lemma 4.5). The Generalized Truth Lemma 5.7 applies to $\mathbb{M}^{\leftrightarrow}_{h,n}$. Then as in Lemma 5.8, we see that $\mathbb{C}_{h,n}(KB4) = \mathbb{M}^{\leftrightarrow}_{h,n}(KB4)$, and $\mathbb{C}_{h,n}(S5) = \mathbb{M}^{\leftrightarrow}_{h,n}(S5)$.

5.2 $K45$ and $KD45$

Proposition 5.12. $\mathbb{M}_{1,1}(K5) \neq \mathbb{C}_{1,1}(K5)$ *and* $\mathbb{M}_{1,1}(K45) \neq \mathbb{C}_{1,1}(K45)$.

Proof We follow the proof of Proposition 5.11. Again we set $\alpha = p \wedge \Diamond p \wedge \Diamond\neg p \wedge \Box(p \vee \neg p)$, and $\beta = \neg p \wedge \Box\mathsf{F}$; so $\alpha \xrightarrow{m} \beta$. Also $\alpha' = p$ and $\beta' = \neg p$. Both α and β are consistent in $S5$, since both are satisfiable in equivalence relations. Also, $\alpha \xrightarrow{m} \alpha$, since $\vdash \alpha \to \Diamond\alpha'$. But working in $K5$, we have $\vdash \Diamond\beta \to \Diamond\Box\mathsf{F}$, and $\vdash \alpha \to \Diamond\mathsf{T}$. So $\vdash \alpha \wedge \Diamond\beta \to (\Diamond\mathsf{T} \wedge \Diamond\Box\mathsf{F})$. Thus $\alpha \wedge \Diamond\beta$ is inconsistent. $\dashv$

Lemma 5.13. *Let $K5 \le L$. Let $\alpha \xrightarrow{c} \beta$ in $\mathbb{C}_{h+1,n}(L)$. If $\gamma \in \mathcal{C}_{h,n}$ and $\vdash \alpha \to \Diamond\gamma$, then $\vdash \beta \to \Diamond\gamma$.*

Proof If not, $\vdash \beta \to \Box\neg\gamma$. So $\vdash \Diamond\beta \to \Diamond\Box\neg\gamma$. But $\alpha \wedge \Diamond\beta$ is consistent, and so is $\alpha \wedge \Diamond\Box\neg\gamma$. Thus $K5$, $\vdash \Diamond\Box\neg\gamma \to \Box\neg\gamma$. so $\alpha \wedge \Box\neg\gamma$ is consistent. By earlier work, $\vdash \alpha \to \Box\neg\gamma$. But this contradicts the consistency of α. ⊣

We define models $\mathbb{N}_{h,n}(L) = (\mathcal{C}_{h,n}(L), \xrightarrow{n})$ by the definition we saw above for the models $\mathbb{M}_{h,n}(L)$, except that we add an additional condition on the accessibility relation $\xrightarrow{n}$:

(4c) whenever $\gamma \in \mathcal{C}_{h,n}$ is such that $\vdash \alpha \to \Diamond\gamma$, then also $\vdash \beta \to \Diamond\gamma$.

Lemma 5.14. *Let $K45 \le L$. If $\alpha \xrightarrow{c} \beta$, then $\alpha \xrightarrow{n} \beta$.*

Proof This follows from Lemmas 5.2, 5.3, and 5.13. ⊣

Lemma 5.15. *Concerning the models $\mathbb{N}_{h,n}(L)$:*

1. *$\mathbb{N}_{h,n}(L)$ is transitive and euclidean.*
2. *If $KT \le L$, then $\mathbb{N}_{h,n}(L)$ is reflexive.*
3. *If $KD \le L$, then $\mathbb{N}_{h,n}(L)$ is serial.*

Proof $\mathbb{N}_{0,n}(L)$ is easily euclidean. We therefore consider only the models $\mathbb{N}_{h+1,n}(L)$. For the transitivity, suppose that $\alpha \xrightarrow{n} \beta$ and $\beta \xrightarrow{n} \gamma$. We show that $\alpha \xrightarrow{n} \gamma$. Since $\beta \xrightarrow{n} \gamma$, we have $\vdash \beta \to \Diamond\gamma'$. And then the fact that $\alpha \xrightarrow{n} \beta$ implies that $\vdash \beta \to \Diamond\gamma'$. Suppose next that $\vdash \alpha \to \Diamond\delta$, where $\delta \in \mathcal{C}_{h,n}(L)$. Then also $\vdash \beta \to \Diamond\delta$, and in addition we have $\vdash \gamma \to \Diamond\delta$. The converse also holds, and we have transitivity.

We turn to the euclidean property. Suppose that $\alpha \xrightarrow{n} \beta$ and $\alpha \xrightarrow{n} \gamma$. We show that $\beta \xrightarrow{n} \gamma$. First, since $\vdash \alpha \to \Diamond\gamma'$ and $\alpha \xrightarrow{n} \beta$, we have $\vdash \beta \to \Diamond\gamma'$. Suppose that $\vdash \beta \to \Diamond\delta$. Then $\vdash \alpha \to \Diamond\delta$, since $\alpha \xrightarrow{n} \beta$. And then since $\alpha \xrightarrow{n} \gamma$, we have $\vdash \gamma \to \Diamond\delta$, as desired.

Turning to the second part of this lemma, if $KT \le L$, then $\vdash \alpha \to \Diamond\alpha'$ for all $\alpha \in \mathcal{C}_{h,n}(L)$. Consequently, $\mathbb{N}_{h,n}(L)$ is reflexive.

Finally, suppose that $KD \le L$. Let $\alpha \in \mathcal{C}_{h,n}(L)$. Then $\vdash \alpha \to \Diamond\mathsf{T}$. By the Existence Lemma, let $\beta \in \mathcal{C}_{h,n}(L)$ be such that $\alpha \xrightarrow{c} \beta$. Then also $\alpha \xrightarrow{n} \beta$. ⊣

Lemma 5.16. $\mathbb{C}_{h,n}(K45) = \mathbb{N}_{h,n}(K45)$, *and* $\mathbb{C}_{h,n}(KD45) = \mathbb{N}_{h,n}(KD45)$.

Proof As in Lemma 5.8: We use Lemmas 5.7, 5.14, and 5.15. ⊣

5.3 $K5$ and $KD5$

It is worth mentioning that the $\mathbb{C}$ models cannot be used to obtain the standard completeness results for these logics.

Proposition 5.17. $\mathbb{C}_{1,1}(K5)$ *and* $\mathbb{C}_{1,1}(KD5)$ *are not euclidean.*

Proof In the notation of Example 2.1, we have $\alpha_5 \xrightarrow{c} \alpha_6$ and $\alpha_5 \xrightarrow{c} \alpha_8$. (The easiest way to see these is to observe that $\alpha_5 \wedge \Diamond\alpha_6$ and $\alpha_5 \wedge \Diamond\alpha_8$ are satisfiable on euclidean serial models.) But $\alpha_8 \not\xrightarrow{c} \alpha_6$. This is because $\vdash \alpha_6 \to \Box\neg p$, so in $K5$, $\vdash \Diamond\alpha_6 \to \Box\neg p$. And $\vdash \alpha_8 \to \Diamond p$. So $\alpha_8 \wedge \Diamond\alpha_6$ is not consistent in $K5$. ⊣

5.4 $K4McK$

McK is the logic generated over K by the McKinsey axioms $\Box\Diamond\varphi \to \Diamond\Box\varphi$. The logic $S4.1$, $S4$ with the McKinsey axioms was shown to have the finite model property by Segerberg [11]. We shall consider $K4McK$ in this section, but our work applies to $S4.1$ with only small changes. Our result is that $K4McK$ is sound for the class C_{McK} of transitive frames with the property that every $x \in F$ has a successor y which has at most one successor.

Lemma 5.18. *For $h \geq 2$, the frame underlying $\mathbb{M}_{h,n}(K4McK)$ belongs to C_{McK}.*

Proof Write $\mathbb{M}$ for the model $\mathbb{M}_{h,n}(K4McK)$, and we write M for its universe, $\mathcal{C}_{h,n}(K4McK)$. $\mathbb{M}$ is transitive by Lemma 5.5, part (1). We therefore show that every $\alpha \in M$ has a successor β which itself has at most one successor. We argue by contradiction.

For each $\gamma \in M$, let

$$\gamma^{\xrightarrow{m}} \quad = \quad \{\delta : \gamma \xrightarrow{m} \delta\}.$$

We claim that each set $\gamma^{\xrightarrow{m}}$ is non-empty. To see this, we recall that $KD \leq K4McK$ and use Lemma 5.5, part (3). (To see that $KD \leq K4McK$, note that in K, $\vdash \Box\Diamond\mathsf{F} \to \Box\mathsf{F}$. In $KMcK$, $\vdash \Box\Diamond\mathsf{F} \to \Diamond\mathsf{T}$. So we then have in $KMcK$ that $\vdash \neg\Box\Diamond\mathsf{F}$; in particular $\vdash \Diamond\mathsf{T}$.)

Towards a contradiction, let $\alpha \in M$ have the property that no element of $\alpha^{\xrightarrow{m}}$ has just one successor. By transitivity, $\alpha^{\xrightarrow{m}}$ is closed under $\xrightarrow{c}$. By finiteness, there is a non-empty set $S \subseteq \alpha^{\xrightarrow{m}}$ which is $\subseteq$-minimal with respect to this closure property. For each $\gamma \in S$, $\gamma^{\xrightarrow{m}} = S$ by minimality. And since $S \neq \emptyset$, our overall assumption in this paragraph implies that there are distinct elements in S, say $\gamma \neq \delta$. Now γ and δ must disagree on some atomic proposition: otherwise they are bisimilar in the model, and hence equal by Lemma 5.6. Suppose that $\gamma \models p$ and $\delta \models \neg p$. Considering S, we see that $\gamma \models \Box\Diamond p$. By Lemma 5.7 and the assumption that $h \geq 2$, in K we have $\vdash \gamma \to \Box\Diamond p$. In $KMcK$, $\vdash \gamma \to \Diamond\Box p$. By the Existence Lemma, there is some $\epsilon \in M$ such that $\epsilon \models \Box p$ and $\delta \xrightarrow{c} \epsilon$ (so also $\delta \xrightarrow{m} \epsilon$). By minimality, $\epsilon^{\xrightarrow{m}} = S$, and so $\epsilon \xrightarrow{c} \delta$. But now $\delta \models p$. This contradicts our earlier fact that $\delta \models \neg p$. $\dashv$

We deduce that $\mathbb{C}_{h,n}(K4McK) = \mathbb{M}_{h,n}(K4McK)$ for $h \geq 2$. Also, we see that $K4McK$ is complete for models whose underlying frame belongs to C_{McK}.

5.5 $K\Box^*$

The next logical system we study is called $K\Box^*$. It is K with the normality axioms and necessitation rules for both $\Box$ and $\Box^*$, together with the following two axiom schemes:

Mix	$\Box^*\varphi \to (\varphi \wedge \Box\Box^*\varphi)$
Induction	$(\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi$

These are the versions of the Segerberg axioms for PDL tuned to this simpler system. This system $K\Box^*$ is sound when $\Box^*$ is interpreted by the reflexive-transitive closure of $\Box$; The rest of this section is devoted to a proof of completeness of $K\Box^*$.

Before this, we need a small point on the multimodal versions of our results. In general, working with more than one modal operator is straightforward in our approach. More precisely, it is straightforward to define the canonical formulas and the analogs of the $\mathbb{C}_{h,n}$ models in the multimodal setting. But it always takes extra work to deal with logics that incorporate non-trivial interaction axioms. This should be no surprise, since even in the monomodal setting one needs special work for the axioms.

We build $\mathbb{C}_{h,n}(K\Box^*)$ the same way we built $\mathbb{C}_{h,n}$ except that we take $\Box$ and $\Box^*$ to be independent modalities. That is, we change (4) to

$$(\bigwedge_{\psi\in R}\Diamond\psi)\ \wedge\ (\Box\bigvee R)\ \wedge\ (\bigwedge_{\psi\in S}\Diamond^*\psi)\ \wedge\ (\Box^*\bigvee S)\ \wedge\ (\bigwedge T)\ \wedge\ (\bigwedge_{p_i\notin T}\neg p_i) \tag{5}$$

But $\mathbb{C}_{h,n}(K\Box^*)$ is the set of formulas of this form which are consistent in $K\Box^*$. We define the *height* of a formula $\varphi \in \mathcal{L}(\Box^*)$ by the same induction as earlier, except we also say that $ht(\Box^*\varphi) = 1 + ht(\varphi)$. The analogs of Lemmas 2.6 and 2.7 hold.

Lemma 5.19. *Let $\alpha, \beta \in \mathcal{C}_{h,n}(K\Box^*)$ and $\Diamond^*\varphi \in \mathcal{L}_{h,n}$. Suppose that $\alpha \xrightarrow{c} \beta$ and $\vdash \beta \to \Diamond^*\varphi$. Then $\vdash \alpha \to \Diamond^*\varphi$ as well.*

Proof If not, then $\vdash \alpha \to \Box^*\neg\varphi$. We also have $\vdash \Diamond\beta \to \Diamond\Diamond^*\varphi$. Since $\alpha \wedge \Diamond\beta$ is consistent, its consequence $\Box^*\neg\varphi \wedge \Diamond\Diamond^*\varphi$ is also consistent. But by Mix, $\vdash \Diamond\Diamond^*\varphi \to \Diamond^*\varphi$. Thus $\Box^*\neg\varphi \wedge \Diamond^*\varphi$ is consistent, and this is absurd. ⊣

We say that a set X is *closed under* $\xrightarrow{c}$ if whenever $\alpha \in X$ and $\alpha \xrightarrow{c} \beta$, then $\beta \in X$ as well.

Lemma 5.20. *Let $X \subseteq \mathcal{C}_{h,n}(K\Box^*)$ be closed under $\xrightarrow{c}$. Then $\vdash \bigvee X \to \Box^*\bigvee X$. Moreover, for all $\alpha \in X$, $\vdash \alpha \to \Box^*\bigvee X$.*

Proof We check that $\vdash \bigvee X \to \Box\bigvee X$. (Then it follows from the induction axiom that $\vdash \bigvee X \to \Box^*\bigvee X$.) If not, $\bigvee X \wedge \Diamond\neg\bigvee X$ is consistent. Then for some $\alpha \in X$, $\alpha \wedge \Diamond\neg\bigvee X$ is consistent. Note that $\neg\bigvee X \in \mathcal{L}_{h,n}$, so we apply the Existence Lemma: let β be such that $\alpha \xrightarrow{c} \beta$ and $\vdash \beta \to \neg\bigvee X$. Then $\beta \notin X$. But then we see that X is not closed under $\xrightarrow{c}$, a contradiction. Finally, for all $\alpha \in X$, $\vdash \alpha \to \bigvee X$. So $\vdash \alpha \to \Box^*\bigvee X$ as well. ⊣

Lemma 5.21 (Truth Lemma for $\mathbb{C}_{h,n}(K\Box^*)$)**.** *For all $\alpha \in \mathcal{C}_{h,n}(K\Box^*)$ and all $\psi \in \mathcal{L}_{h,n}$,*

$$(\mathbb{C}_{h,n}(K\Box^*), \alpha) \models \psi \quad \textit{iff} \quad \vdash \alpha \to \psi \textit{ in } K\Box^*.$$

Proof By induction on ψ. The step for $\Diamond\psi$ is the same as in Lemma 4.4. We only give the induction step for $\Diamond^*\psi$ formulas.

Let $\Diamond^*\psi \in \mathcal{L}_{h,n}$. First, suppose that $(\mathbb{C}_{h,n}(K\Box^*), \alpha) \models \Diamond^*\psi$. We check by induction on the length l of the shortest path witnessing this that $\vdash \alpha \to \Diamond^*\psi$. For $l = 0$, we have $\alpha \models \psi$, and the induction hypothesis shows $\vdash \alpha \to \psi$. And by Mix, $\vdash \psi \to \Diamond^*\psi$. Assume the result for l, and suppose that $\alpha \models \Diamond^*\psi$ via a path of length $l + 1$. Let $\alpha \xrightarrow{c} \beta$ and $\beta \models \Diamond^*\psi$. By induction hypothesis, $\vdash \beta \to \Diamond^*\psi$. By Lemma 5.19, $\vdash \alpha \to \Diamond^*\psi$.

In the other direction, assume that $\vdash \alpha \to \Diamond^*\psi$. Let X be the set of $\beta \in \mathcal{C}_{h,n}(K\Box^*)$ such that there is *no* path in the model from β to any γ such that $\gamma \models \psi$. Note that X is closed under $\xrightarrow{c}$. We must show that $\alpha \notin X$, for then our semantics tells us that $\alpha \models \Diamond^*\psi$. If $\alpha \in X$, then by Lemma 5.20, $\vdash \alpha \to \Box^*\bigvee X$. For $\beta \in X$, $\vdash \beta \to \neg\psi$. So $\vdash \bigvee X \to \neg\psi$, and it follows that $\vdash \alpha \to \Box^*\neg\psi$. And we have a contradiction to $\vdash \alpha \to \Diamond^*\psi$. ⊣

Theorem 5.22. *$K\Box^*$ is complete and decidable.*

6 Two modifications

In this section, we present two further completeness results. These have the property that the models $\mathbb{C}_{h,n}(L)$ are not directly usable; one must modify them in some way or other. This section may be omitted without loss of continuity.

6.1 $\Box\varphi \leftrightarrow \Diamond\varphi$

We consider Tr, the logic K together with the axioms $\Box\varphi \leftrightarrow \Diamond\varphi$. The corresponding frame condition is that every point has exactly one successor; that is, the frame is the graph of a total function. We'll write L in this discussion for this logic. As it happens, $\mathbb{C}_{h,n}(L)$ is never a function. The situation is simple enough that we can give an explicit description of the model $\mathbb{C}_{h,n}(L)$. Recall the sets SD_n of state descriptions of order n; see equation (1). The set $\mathcal{C}_{h,n}(L)$ of worlds of the model $\mathbb{C}_{h,n}(L)$ is isomorphic to the set of sequences of length h of elements of SD_n. We write such a sequence as

$$s \quad = \quad (\widehat{S_1}, \ldots, \widehat{S_h}).$$

The atomic formulas true at s are those implied by S_1. Given two such sequences, say the one above and also $t = (\widehat{T_1}, \ldots, \widehat{T_h})$, we say that $s \to t$ iff $S_2 = T_1, \ldots, S_h = T_{h-1}$. In this way, each point has 2^n successors. The overall model is thus not a function. The isomorphism $i : W \to \mathcal{C}_{h,n}(L)$ is given by

$$i(s) \quad = \quad \widehat{S_1} \wedge \Diamond^+(\widehat{S_2} \wedge \Diamond^+(\cdots \wedge \Diamond^+\widehat{S_n}))$$

where $\Diamond^+\varphi$ abbreviates $\Diamond\varphi \wedge \Box\varphi$.

We can, however obtain a completeness result by modifying the model. Let D be any subset of $\mathcal{C}_{h,n}(L)$ which is a total function. (D may be obtained by starting with any node, following the arrows in any way whatsoever, and stopping on a repeated node.) An induction on i shows that for $1 \leq i \leq h$ and $\varphi \in \mathcal{L}_{i,n}$ and all s and t such that $S_1 = T_1, \ldots, S_i = T_i$, we have $(\mathbb{C}_{h,n}(L), s) \models \varphi$ iff $(\mathbb{C}_{h,n}(L), t) \models \varphi$. Then an induction on $\varphi \in \mathcal{L}_{h,n}$ shows that for $s \in W$, $(\mathbb{C}_{h,n}(L), s) \models \varphi$ iff $(W, s) \models \varphi$. It follows that for $\alpha \in \mathcal{C}_{h,n}$, $(W, i^{-1}(\alpha)) \models \alpha$. In particular, every L-consistent α has a functional model.

6.2 The Löb logic KL

Recall that the Löb axioms are those of the form $\Box(\Box\varphi \to \varphi) \to \Box\varphi$, and KL is K together with all of these axioms. Recall also the standard fact that $K4 \leq KL$. (It would have been nice to find a semantic proof of this by studying $\mathbb{C}_{h,n}(KL)$ directly, but we were not able to do this.)

In this section, we study a variant $\mathbb{D}_{h,n}(L) = (\mathcal{C}_{h,n}(L), \xrightarrow{d})$ of $\mathbb{C}_{h,n}(L)$, and we show that this model is transitive and converse wellfounded. This gives the completeness and decidability results for the logic KL on finite transitive tree models.

We define $\alpha \xrightarrow{d} \beta$ if $\alpha \xrightarrow{c} \beta$, and for some $\varphi \in \mathcal{L}_{h,n}$, $\vdash \alpha \to \Diamond\varphi$ and $\vdash \beta \to \Box\neg\varphi$. The valuation on the model is the obvious one.

We also define $\mathbb{N}^*_{h,n}(L) = (\mathcal{C}_{h,n}(L), \to_{N^*})$ by the same definition of $\mathbb{M}_{h,n}(L)$, except that we also add

$$(3c) \qquad \text{there is some } \gamma \in \mathcal{C}_{h,n} \text{ such that } \vdash \alpha \to \Diamond\gamma, \text{ but } \vdash \beta \to \Box\neg\gamma.$$

Since $K4 \leq L$ we have that if $\alpha \xrightarrow{d} \beta$, then $\alpha \to_{N^*} \beta$ (see also Lemma 5.4).

Further, $\mathbb{N}^*_{h,n}(L)$ is transitive. To see this, suppose that $\alpha \to_{N^*} \beta \to_{N^*} \gamma$, and let γ and δ be such that $\vdash \alpha \to \Diamond\gamma$ and $\vdash \beta \to \Box\neg\gamma$. Then we must have $\vdash \gamma \to \Box\neg\gamma$; for if not, then $\vdash \gamma \to \Diamond\gamma$, so $\vdash \beta \to \Diamond\gamma$ by Lemma 5.4.

Continuing, we claim that $\mathbb{N}^*_{h,n}(L)$ is converse wellfounded. That is, there are no infinite paths

$$\alpha_0 \to_{N^*} \alpha_1 \to_{N^*} \alpha_2 \to_{N^*} \cdots$$

in the model. For suppose we had such a path. For each i, let $\gamma_i \in \mathcal{C}_{h,n}$ be such that $\vdash \alpha_i \to \Diamond\gamma_i$ and $\vdash \alpha_{i+1} \to \Box\neg\gamma_i$. Since $\mathcal{L}_{h,n}$ is finite, we can fix some γ^* such that for infinitely many n, $\gamma_i = \gamma^*$. But then for some $i < j$ we have $\vdash \alpha_i \to \Diamond\gamma^*$ and $\vdash \alpha_j \to \Diamond\gamma^*$. However, by an inductive argument using Lemma 5.3 backwards from j we also see that $\vdash \alpha_{i+1} \to \Diamond\gamma^*$. This contradicts the consistency of α_{i+1}. This establishes our claim.

Lemma 6.1 (Truth Lemma for $\mathbb{N}^*_{h,n}(L)$)**.** *Let $KL \leq L$. For all $\alpha \in \mathcal{C}_{h,n}(L)$ and all $\psi \in \mathcal{L}_{h,n}$, $(\mathbb{N}^*_{h,n}(L), \alpha) \models \psi$ iff $\vdash \alpha \to \psi$ in K.*

Proof We only give the induction step for $\Diamond\psi$ formulas. Let $\Diamond\psi \in \mathcal{L}_{h,n}$. Assume first that $(\mathbb{N}^*_{h,n}(L), \alpha) \models \Diamond\psi$. Then just as in Lemma 4.4 we see that $\vdash \alpha \to \Diamond\psi$.

In the other direction, suppose that $\vdash \alpha \to \Diamond\psi$. Then by the Löb axioms, $\vdash \alpha \to \Diamond(\psi \wedge \Box\neg\psi)$. Note that $\psi \wedge \Box\neg\psi \in \mathcal{L}_{h,n}$ as well. By the Existence Lemma 4.3, let $\beta \in \mathcal{C}_{h,n}(L)$ be such that $\alpha \xrightarrow{c} \beta$ and $\vdash \beta \to (\psi \wedge \Box\neg\psi)$. The last condition insures that $\alpha \xrightarrow{d} \beta$. So $\alpha \to_{N^*} \beta$, and we have $(\mathbb{N}^*_{h,n}(L), \alpha) \models \Diamond\psi$. ⊣

Theorem 6.2. *Concerning KL:*

1. *$\mathbb{N}^*_{h,n}(KL) = \mathbb{D}_{h,n}(KL)$.*
2. *KL is complete for finite transitive, converse wellfounded models.*

Proof The first part is similar to arguments that we have seen. The two models have the same set of nodes. If $\alpha \xrightarrow{d} \beta$, then we know that $\alpha \to_{N^*} \beta$. Conversely, if $\alpha \to_{N^*} \beta$, then by Lemma 6.1, $\alpha \wedge \Diamond\beta$ is satisfiable on some transitive converse wellfounded model, hence it is consistent in KL.

The second follows from the first, as we have seen many times. ⊣

Since we are interested in the models $\mathbb{C}_{h,n}(L)$, it is worth noting that for the logic at hand we do have transitivity.

Theorem 6.3. *$\mathbb{C}_{h,n}(KL)$ is transitive.*

Proof Let $\approx$ be the relation on $\mathbb{C}_{h,n}(KL)$ defined by

$$\alpha \approx \beta \quad \text{iff} \quad \text{for all } \delta \in \mathcal{C}_{h-1,n}, \vdash \alpha \to \delta \text{ iff } \vdash \beta \to \delta$$

Note in particular that if $\alpha \approx \beta$ that α and β are the same except for their atomic conjunct. As a consequence, $\alpha' = \beta'$.

Further, note that $\alpha \xrightarrow{c} \beta$ iff either $\alpha \to_{N^*} \beta$ or $\alpha \approx \beta$. Turning to the transitivity, suppose that $\alpha \xrightarrow{c} \beta \xrightarrow{c} \gamma$. There are four cases. Once case uses the transitivity of $\to_{N^*}$, and another the transitivity of $\approx$. The remaining two cases are similar to each other; here is one: suppose $\alpha \to_{N^*} \beta \approx \gamma$. Then $\vdash \alpha \to \Diamond\beta'$, so as $\Diamond\beta' = \Diamond\gamma'$, we have $\vdash \alpha \to \Diamond\gamma'$. Also, if $\vdash \gamma \to \Diamond\delta$, then also $\vdash \beta \to \Diamond\delta$ (since $\beta \approx \gamma$). And then also $\vdash \alpha \to \Diamond\delta$ (since $\alpha \xrightarrow{m} \beta$). Finally, if δ is such that $\vdash \alpha \to \Diamond\delta$ but $\vdash \beta \to \Box\neg\delta$, then again we have $\vdash \gamma \to \Box\neg\delta$. $\dashv$

As it happens, $\mathbb{C}_{1,1}(KL) = \mathbb{C}_{1,1}(K4)$. This model is shown in the bottom right corner of Figure 1. Some work with a computer program shows that $\mathbb{C}_{2,1}(KL)$ has 170 nodes.

7 Comparison with other work

One of the main points of this paper is the study of the models $\mathbb{C}_{h,n}(L)$ and the application of those models to completeness proofs. As a summary/conclusion to this paper, we mention the relation of our work with that of others, especially the original paper of Fine [4] and also the approach via filtration.

7.1 Comparison with filtration of the canonical model

Recall that $\mathcal{L}_{h,n}$ is the set of modal formulas of height $\leq h$ and of order $\leq n$. Let Let $\mathsf{Can}(L)$ be the canonical model of a logic L. This is the model whose worlds are the maximal consistent sets in L, whose valuation is $v(p) = \{x : p \in x\}$, and whose accessibility relation is $x \to y$ iff whenever $\varphi \in y$, $\Diamond\varphi \in x$. For each h and n, each $x \in \mathsf{Can}(L)$ contains a unique element $\alpha_x \in \mathcal{C}_{h,n}$. The set $\mathcal{L}_{h,n}$ induces an equivalence relation $\equiv$ on $\mathsf{Can}(L)$, where $x \equiv y$ iff $\alpha_x = \alpha_y$. This is the same relation as $x \equiv y$ iff $x \cap \mathcal{L}_{h,n} = y \cap \mathcal{L}_{h,n}$. We recall also the Truth Lemma for $\mathsf{Can}(L)$: $W \models \varphi$ iff $\varphi \in W$.

For each $\alpha \in \mathcal{C}_{h,n}(L)$, $\mathsf{Can}(L)$ contains $Th_{\mathbb{C}}(\alpha)$, the set of formulas satisfied by α in the model $\mathbb{C}_{h,n}$. Note that $\alpha_{Th(\alpha)} = \alpha$, and $Th(\alpha_x) \equiv x$. (But in general, we do not have $Th(\alpha_x) = x$.)

Recall also that in view of the Truth Lemma, a *filtration* on the canonical model through $\mathcal{L}_{h,n}$ is a relation R on $\mathsf{Can}(L)/\equiv$ such that

1. if $x \to y$, then $[x]R[y]$;
2. if $[x]R[y]$, $\Diamond\varphi \in \mathcal{L}_{h,n}$ and $\varphi \in y$, then also $\Diamond\varphi \in x$.

The smallest filtration is R_{min}, given by $[x]R[y]$ iff $x' \to y'$ for some $x' \equiv x$ and $y' \equiv y$. The largest is R_{max} is given by $[x]R_{max}[y]$ iff whenever $\Diamond\varphi \in \mathcal{L}_{h,n}$ and $\varphi \in y$, $\Diamond\varphi \in x$.

Theorem 7.1. *There is a one-to-one correspondence between filtrations of* $\mathsf{Can}(L)$ *through* $\mathcal{L}_{h,n}$ *and suitable relations* $\rightsquigarrow$ *on* $\mathcal{C}_{h,n}(L)$. *The correspondence associates to a filtration* R *the suitable relation* $\rightsquigarrow_R$ *given by*

$$\alpha \rightsquigarrow_R \beta \quad \textit{iff} \quad [Th(\alpha)]R[Th(\beta)].$$

In the other direction, we associate to a suitable relation $\rightsquigarrow$ the filtration $R_{\rightsquigarrow}$ given by

$$[x]R_{\rightsquigarrow}[y] \quad \textit{iff} \quad \alpha_x \rightsquigarrow \alpha_y$$

Each of these is monotone with respect to inclusion of relations. Moreover,

1. *The minimal filtration of $\mathcal{L}_{h,n}$ corresponds to the accessibility relation $\xrightarrow{c}$ of $\mathbb{C}_{h,n}(L)$.*
2. *The largest filtration on $\mathcal{L}_{h,n}$ corresponds to the suitable relation $\rightsquigarrow$ given by $\alpha \rightsquigarrow \beta$ iff $\vdash \alpha \to \beta'$ in L.*

Proof There are a few things to check here. We begin with the verification that if R is a filtration, then we check that $\rightsquigarrow_R$ is suitable. Note that if $\alpha \xrightarrow{c} \beta$, then $Th(\alpha) \to Th(\beta)$ in $\mathsf{Can}(L)$. And then by the first filtration condition, we have $[Th(\alpha)]R[Th(\beta)]$. Therefore $\alpha \rightsquigarrow_R \beta$. And if $\alpha \rightsquigarrow_R \beta$, then we show $\alpha \wedge \Diamond\beta'$ is consistent in K; it follows that $\vdash \alpha \to \Diamond\beta'$ in K. Note that in the filtration $\mathsf{Can}(L)/R$, $[Th(\alpha)] \models \alpha$ and $[Th(\beta)] \models \beta'$. These hold by the Filtration Lemma and the Truth Lemma of the canonical model.

In the other direction, assume that $\rightsquigarrow$ is a suitable relation on $\mathcal{C}_{h,n}(L)$. We check that $R_{\rightsquigarrow}$ is a filtration of $\mathsf{Can}(L)$. Suppose that $x \to y$ in $\mathsf{Can}(L)$. Then in the canonical model, $x \models \alpha_x \wedge \Diamond\alpha_y$. So $\alpha_x \wedge \Diamond\alpha_y \in x$. Therefore $\alpha_x \wedge \Diamond\alpha_y$ is consistent in L; i.e., $\alpha_x \xrightarrow{c} \alpha_y$. We then have $\alpha_x \rightsquigarrow \alpha_y$. Thus $[x]R[y]$. And suppose that $[x]R[y]$, $\Diamond\varphi \in \mathcal{L}_{h,n}$ and $\psi \in y$. In $\mathsf{Can}(L)$, $y \models \psi$. By the Filtration Theorem, $[y] \models \varphi$. So$[x] \models \Diamond\varphi$, and thus $\Diamond\varphi \in x$ as desired.

To check that the two correspondences are inverses:

$$\begin{array}{rll} \alpha \rightsquigarrow_{R_{\rightsquigarrow}} \beta & \text{iff} & [Th(\alpha)]R_{\rightsquigarrow}[Th(\beta)] \\ & \text{iff} & \alpha_{Th(\alpha)} \rightsquigarrow \beta_{Th(\beta)} \\ & \text{iff} & \alpha \rightsquigarrow \beta \end{array}$$

and also

$$\begin{array}{rll} [x]R_{\rightsquigarrow_R}[y] & \text{iff} & \alpha_x \rightsquigarrow_R \alpha_y \\ & \text{iff} & [Th(\alpha_x)]R[Th(\alpha_y)] \\ & \text{iff} & [x]R[y] \end{array}$$

The monotonicity point in our theorem means that if $R_1 \subseteq R_2$ are filtrations, then $\rightsquigarrow_{R_1} \subseteq \rightsquigarrow_{R_2}$. This is immediate from the definition, as is the parallel point in the other direction.

We next turn to the additional assertions. Now

$$\begin{array}{rll} \alpha \rightsquigarrow_{R_{min}} \beta & \text{iff} & [Th(\alpha)]R_{min}[Th(\beta)] \\ & \text{iff} & (\exists x', y')x' \equiv Th(\alpha), y' \equiv Th(\beta), x' \to y' \\ & \text{iff} & \alpha_{x'} \wedge \Diamond\beta_{y'} \text{ is consistent} \\ & \text{iff} & \alpha \wedge \Diamond\beta \text{ is consistent} \\ & \text{iff} & \alpha \xrightarrow{c} \beta \end{array}$$

Finally,

$$\begin{array}{rll} \alpha \rightsquigarrow_{R_{max}} \beta & \text{iff} & [Th(\alpha)]R_{max}[Th(\beta)] \\ & \text{iff} & \Diamond\varphi \in \mathcal{L}_{h,n} \text{ and } (\mathcal{C}_{h,n}, \beta) \models \varphi \text{ imply } (\mathcal{C}_{h,n}, \alpha) \models \Diamond\varphi \\ & \text{iff} & \Diamond\varphi \in \mathcal{L}_{h,n} \text{ and } \vdash \beta' \to \varphi \text{ imply } (\mathcal{C}_{h,n}, \alpha) \models \Diamond\varphi \\ & \text{iff} & \vdash \alpha \to \Diamond\beta' \end{array}$$

$\dashv$

In view of the results of this paper on $\mathcal{C}_{h,n}(L)$, one might at first glance think that we could profitably consider the models $\mathbb{D}_{h,n}(L) := (\mathcal{C}_{h,n}(L), \rightsquigarrow_{R_{max}})$ obtained by the largest filtration. However, these models are usually not interesting. For example, it is easy to check that $\mathbb{D}_{h,n}(K4)$ is has more edges than $\mathbb{C}_{h,n}(K4)$; specifically, $\widehat{\emptyset} \wedge p$ has outgoing arrows in $\mathbb{D}_{h,n}(K4)$. This means that $\mathbb{D}_{h,n}(K4)$ is not transitive. The upshot is that the models $\mathbb{C}_{h,n}(K4)$ are more appropriate vehicles for completeness results.

The canonical model as a limit Since we have been discussing the canonical model, it might be useful to mention a sense in which the canonical model may be regarded as the limit of the models $\mathbb{C}_{h,n}(L)$ that have been the main objects of study in this paper. It is not entirely clear how to make this precise.

Fix a number n, and let $\mathcal{L}_n = \bigcup_h \mathcal{L}_{h,n}$. This is the set of formulas on the first n atomic propositions. Let $\mathsf{Can}_n(L)$ be the set of consistent formulas in $\mathcal{L}_n$, made into a Kripke model in the expected way. All of the work we do in this section is independent of n, so we shall drop it from the notation.

Recall that an n-bisimulation between Kripke models $\mathbb{A}$ and $\mathbb{B}$ is a sequence $Z_0 \supseteq Z_1 \cdots \supseteq Z_n$ such that

1. If aZ_0b, then a and b agree on atomic propositions.
2. If $aZ_{i+1}b$ and $a \to a^*$, then there is some b^* such that $b \to b^*$ and $a^*Z_ib^*$.
3. If $aZ_{i+1}b$ and $b \to b^*$, then there is some a^* such that and $a^*Z_ib^*$.

For more on this, see, e.g., Blackburn, de Rijke, and Venema [3], pp. 74–75. We will also speak of an ω-bisimulation as an infinite sequence with the properties above for all i. It is well-known that this is weaker than a bisimulation.

Proposition 7.2. *The following sequence*

$$\widehat{Z}^h \quad = \quad Z_0 \supseteq Z_1 \cdots \supseteq Z_h$$

is the largest h-bisimulation between $\mathbb{C}_{h+1,n}(L)$ and $\mathbb{C}_{h,n}(L)$: $\alpha Z_i \beta$ iff α and β satisfy the same formula in $\mathcal{C}_{i,n}$. In terms of derivatives, $\alpha^{(h+1-i)} = \beta^{(h-i)}$.

Proof It is clear that if $\alpha Z_0 \beta$, then α and β agree on atomic propositions. We check one of the zig-zag conditions. Suppose that α and β agree on the same canonical formula of height $i+1$, say γ, and also that $\alpha \xrightarrow{c} \alpha^*$. Let δ be the element of $\mathcal{C}_{i,n}$ satisfied by α^*. Then $\vdash \gamma \to \Diamond\delta$, since $\gamma \wedge \Diamond\delta$ is consistent. From this, $\beta \models \Diamond\delta$. So for some β^* such that $\beta \xrightarrow{c} \beta^*$, $\beta^* \models \delta$. This means that δ is the element of $\mathcal{C}_{h,n}(L)$ satisfied by β.

For the uniqueness, recall that if two points in any models are related by an h-bisimulation, then they satisfy the same modal formulas of height at most h. For $0 \leq i \leq h$, the modal formula of height i satisfied by α in $\mathbb{C}_{h+1,n}$ is $\alpha^{(h+1-i)}$ (this is the $(h+1-i)$-th derivative of α, and the modal formula of height i satisfied by β in $\mathbb{C}_{h,n}$ is $\beta^{(h-i)}$. ⊣

Consider the infinite sequence along the top below:

$$\begin{tikzcd}
\mathbb{C}_{0,n}(L) & \mathbb{C}_{1,n}(L) \arrow[l, "\widehat{Z}_0"'] & \mathbb{C}_{2,n}(L) \arrow[l, "\widehat{Z}_1"'] & \cdots & \mathbb{C}_{h,n}(L) & \mathbb{C}_{h+1,n}(L) \arrow[l, "\widehat{Z}_h"'] & \cdots \\
 & & & & \mathsf{Can}_n(L) \arrow[u, "\pi^h"] \arrow[ur, "\pi^{h+1}"'] & &
\end{tikzcd}$$

So the relations on the top line are stronger and stronger with h. (That is, the diagram above is not a diagram in a category in the usual sense, since the relations (morphisms) involve are required to be have *different* properties.) The h-bisimulation π^h is the sequence $Y_0 \supseteq Y_1 \cdots \supseteq Y_h$, where $xY_i\alpha$ iff x and α satisfy the same formula in $\mathcal{C}_{i,n}(L)$. It is not hard to check that for all h, π^h is an h-bisimulation. Moreover, the diagram commutes in the appropriate sense: for all h and all $i \leq h$, $Y_i^h = Z_i^h \circ Y_i^{h+1}$ as a relational composition.

Proposition 7.3. *$\mathsf{Can}_n(L)$ is the limit of the sequence $\mathbb{C}_{h,n}(L) \leftarrow \mathbb{C}_{h+1,n}(L)$ in the sense that if $\mathbb{A} = (A, \rightarrow, v)$ is any L-model and for all h, $\rho^h = Z_0 \supseteq Z_1 \supseteq \cdots Z_h$ is an h-bisimulation between $\mathcal{A}$ and $\mathcal{C}_{h,n}(L)$, then there is a unique ω-bisimulation $R_0 \supseteq R_1 \supseteq \cdots$ such that for all h, $\rho^h = \pi^h \circ R_h$. Moreover, there is a total bisimulation between $\mathbb{A}$ and Can_n.*

Proof (Sketch) The hypotheses imply that for all h, if $aZ_h\alpha$, then $(\mathbb{A}, a) \models \alpha$. It then turns out that the map $d : A \rightarrow \mathsf{Can}_n$ given by $d(a) = \{\varphi : (\mathbb{A}, a) \models \varphi\}$ is the unique bisimulation. The ω-bisimulation between them is defined similarly. $\dashv$

7.2 Comparison with Fine's original treatment

In this section, we return to our discussion from the Introduction concerning Fine's paper [4] on this subject. What we want to do here is to review details on his model $\mathfrak{C}_{h,n}$, the vehicle for proving completeness of $K4$. We show that it is the same as $\mathbb{C}_{h,n}(K4)$.

For any formulas α and β in $\mathcal{C}_{h,n}$, we say $\alpha\, S\, \beta$ if for all $\gamma \in \mathcal{C}_{h-1,n}$, $\vdash \beta \rightarrow \Diamond\gamma$ implies $\vdash \alpha \rightarrow \Diamond\gamma$. Here the provability is in K. A formula $\alpha \in \mathcal{C}_{h,n}$ is $K4$-*suitable* if for $\beta \in \mathcal{C}_{h-1,n}$ and $\gamma \in \mathcal{C}_{h-2,n}$, if $\vdash \alpha \rightarrow \Diamond\beta$ and $\vdash \beta \rightarrow \Diamond\gamma$, then there is some $\delta \in \mathcal{C}_{h-1,n}$ such that $\vdash \alpha \rightarrow \Diamond\delta$, $\delta' = \gamma$, and $\beta\, S\, \delta$.

We define a model $(\mathfrak{C}_{h,n}, R)$ as follows: the worlds of the model are the $K4$-suitable formulas, and the accessibility relation is $\alpha R \beta$ iff $\vdash \alpha \rightarrow \Diamond\beta'$ and $\alpha\, S\, \beta$. The valuation is the obvious one. It is clear that R is transitive. Note that the logical system $K4$ plays no role in any of these definitions.

The following result is neither surprising nor trivial:

Proposition 7.4. $\mathbb{M}_{h,n}(K4) = \mathbb{C}_{h,n}(K4) = \mathfrak{C}_{h,n}$.

Proof First we check that every $\alpha \in \mathcal{C}_{h,n}(K4)$ is $K4$-suitable. For this, we call on Lemma 4 from [4]. That result concerns the property of $\alpha \in \mathcal{C}_{h,n}$ being $K4$-*suited*: if $\beta \in \mathcal{C}_{h-2,n}$ and $\vdash \alpha \rightarrow \Diamond\Diamond\beta$, then for some $\gamma \in \mathcal{C}_{h-1,n}$, $\gamma' = \beta$, and $\vdash \alpha \rightarrow \Diamond\gamma$.

We claim at this point that for all n, every $\alpha \in \mathcal{C}_{h,n}(K4)$ is $K4$-suited. For suppose that $\beta \in \mathcal{C}_{h-2,n}$ and $\vdash \alpha \rightarrow \Diamond\Diamond\beta$. Then in the model $\mathbb{C}_{h,n}(K4)$, $\alpha \models \Diamond\Diamond\beta$. So let $\gamma^\#, \delta^\# \in \mathcal{C}_{h,n}(K4)$ be such that $\alpha \xrightarrow{c} \gamma^\# \xrightarrow{c} \delta^\#$ and $\delta^\# \models \Diamond\beta$. Then the second derivative of $\delta^\#$ is exactly β. We take for γ the first derivative of $\delta^{\#}$. We have $\gamma' = \beta$. Further, by transitivity of the model $\mathbb{C}_{h,n}(K4)$, $\alpha \xrightarrow{c} \delta^{\#}$. So in the model, $\alpha \models \Diamond\gamma$. By the Truth Lemma, $\vdash \alpha \to \Diamond\gamma$.

We also claim that for every $\alpha \in \mathcal{C}_{h,n}(K4)$ there is some $\beta \in \mathcal{C}_{h+1,n}(K4)$ such that $\beta' = \alpha$. To see this, let β be the canonical formula of height h satisfied by α in $\mathbb{C}_{h+1,n}$. This model is transitive, so β is consistent in $K4$. We have $\vdash \beta \to \alpha$ by properties of $\mathcal{C}_{h,n}$, and so indeed $\alpha = \beta'$.

Lemma 4 of [4] tells us that if α is $K4$-suited, then α' is $K4$-suitable. Combined with the results of the last two paragraphs, we see that indeed every $\alpha \in \mathcal{C}_{h,n}(K4)$ is $K4$-suitable.

Theorem 5 in [4] proves that $(\mathfrak{C}_{h,n}, \alpha) \models \alpha$ for all $K4$-suitable α. Further, R is transitive on the model. So we see that each $K4$-suitable α is consistent in $K4$. Thus, the two models $\mathbb{C}_{h,n}(K4)$and $\mathfrak{C}_{h,n}$ have the same worlds. If is immediate from the definitions that the two relations $\xrightarrow{m}$ and R are the same, as are their valuations. $\dashv$

8 Conclusions and open problems

This paper has not presented new results. We take its main contribution to be a demonstration that model constructions can be based on canonical formulas. To put things in a different light, recall that in first-order model theory, one of the basic constructions is the Henkin model: one build a model from a set of first-order sentences. One would think that *finite* models could be built from sentences in certain settings. This idea cannot work out for first-order logic in general, since the logic lacks the finite model property. But for modal logic, or any logic with the finite model property, it is natural to ask for model constructions that based on the syntactic objects and closely related notions (such as the consistency of some formula in some logic). This is what we studied in this paper. The fact that the results could have been obtained otherwise does not detract from a direct development.

After seeing Theorem 7.1 on filtration, one might question this paper. After all, if the specific completeness results are already known and if the particular models happen to be obtainable as special cases of known work, what exactly is gained? There are several responses. Even if one is happier with the classical development than with our re-working of it, it still might be useful to know about the minimal filtration of the canonical model through $\mathcal{L}_{h,n}$. So all of our work could be re-cast as a body of results on that particular topic.

We close the paper by reiterating the open problem from Section 5: is it true that for every logic L such that $K4 \leq L$, $\mathcal{C}_{h,n}(L)$ is transitive for almost all h? This turns out to be true for the extensions of $K4$ in this paper, but the proofs are so different that one doubts that a single result is behind them.

We also think it would be interesting to pursue other aspects of modal logic using the model constructions from this paper. For example, one might hope for a treatment of the computational complexity of various logics based on the models $\mathbb{C}_{h,n}(L)$.

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