Neighborhood Semantics for Modal Logic

Lecture 2

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- Conditional Logic
- Logic of knowledge, evidence and belief
- Coalitional logic

Conditional Logic

One of the earliest applications of neighborhood models is found in David Lewis's seminal book *Counterfactuals*.

Conditionals

1. If it's a square, then it's rectangle.
2. If $A \subseteq B$, then $A \cap B = A$.
3. If you strike the match, it will light.
4. If you had struck the match, it would have lit.

Role of conditionals in mathematical, practical and causal reasoning.

Material conditional: $\varphi \rightarrow \psi$

$\varphi \rightarrow \psi$ is **true** if either the antecedent (φ) is false or the consequent (ψ) is true.

Conditional: $\varphi \mathrel{\Box\!\!\rightarrow} \psi$

Whatever the proper analysis of the contrafactual conditional may be, we may be sure in advance that it cannot be truth-functional; for, obviously ordinary usage demands that some contrafactual conditionals with false antecedents and false consequents be true and that other contrafactual conditionals with false antecedents and false consequents be false

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- If I weighed more than 300 pounds, I would weigh more than 200 pounds.
- If I weighed more than 300 pounds, I would weigh less than 10 pounds.

(MP)	$\varphi \to \psi, \varphi \vdash \psi$
(MT)	$\varphi \to \psi, \neg\psi \vdash \neg\varphi$
(DS)	$\varphi \vee \psi \vdash \neg\varphi \to \psi$

$$
\begin{array}{rl}
\text{(FA)} & \neg\varphi \vdash \varphi \to \psi \\
\text{(TC)} & \psi \vdash \varphi \to \psi \\
\text{(C)} & \varphi \to \psi \vdash \neg\psi \to \neg\varphi \\
\text{(Mon)} & \varphi \to \psi \vdash (\varphi \wedge \chi) \to \psi \\
\text{(Trans)} & \varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi
\end{array}
$$

$\neg\varphi \vdash \varphi \rightarrow \psi$

Beijing is not in Maryland. ?? So, if Beijing is in Maryland, then Trump is a Democrat.

$\neg\varphi \vdash \varphi \to \psi$

Beijing is not in Maryland. ?? So, if Beijing is in Maryland, then Trump is a Democrat.

$\psi \vdash \varphi \to \psi$

Eric was in Maryalnd this morning. ?? So, if Eric was in Beijing this morning, then Eric was in Maryland this morning.

$\varphi \rightarrow \psi \vdash (\varphi \wedge \chi) \rightarrow \psi$

If I put sugar in my coffee, then it will taste good. ?? So, if I put sugar and gasoline in my coffee, then it will taste good.

If this match is struck, then it will light. ?? So, if this match is soaked overnight and struck, then it will light.

$\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$

If I quit my job, I won't be able to afford my apartment. But if I win 10 million dollars, I will quit my job. ?? So, if I win 10 million dollars, I won't be able to afford my apartment.

Sphere Models

A **set of spheres** is a subset space $\langle W, \mathcal{S}\rangle$, where

- $\mathcal{S}$ is *nested*: For all $S, T \in \mathcal{S}$, either $S \subseteq T$ or $T \subseteq S$.
- $\mathcal{S}$ is *closed under unions*: If $\{S_i \mid i \in I\} \subseteq \mathcal{S}$ for some index set I, then $\bigcup_{i\in I} S_i \in \mathcal{S}$.
- $\mathcal{S}$ is *closed under intersections*: If $\{S_i \mid i \in I\} \subseteq \mathcal{S}$ for some index set I, then $\bigcap_{i\in I} S_i \in \mathcal{S}$.

We say that a system of spheres $\langle W, \mathcal{S}\rangle$ is **centered** on $w \in W$ provided that $\{w\} \in S$.

Sphere Frames/Models

A **sphere frame** is a neighborhood frame $\langle W, N \rangle$, where $W \neq \emptyset$ and for all $w \in W$, $\langle W, N(w) \rangle$ is a set of spheres. We say that $\langle W, N \rangle$ is **centered** provided that for all $w \in W$, $N(w)$ is centered on w.

A **sphere model** is a tuple $\langle W, N, V \rangle$ where $\langle W, N \rangle$ is a sphere frame and $V : \mathrm{At} \to \wp(W)$ is a valuation function.

Conditional Logic

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid (\varphi \mathrel{\Box\!\!\rightarrow} \psi)$$

The intended interpretation of $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ is "if φ, then ψ".

Conditional Logic

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid (\varphi \mathrel{\Box\!\!\rightarrow} \psi)$$

The intended interpretation of $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ is "if φ, then ψ".

$\mathcal{M}, w \models \varphi \mathrel{\Box\!\!\rightarrow} \psi$ iff either $\bigcup N(w) \cap [\![\varphi]\!]_{\mathcal{M}} = \emptyset$ or there is a $S \in N(w)$ such that $[\![\varphi]\!]_{\mathcal{M}} \cap S \neq \emptyset$ and $[\![\varphi]\!]_{\mathcal{M}} \cap S \subseteq [\![\psi]\!]_{\mathcal{M}}$.

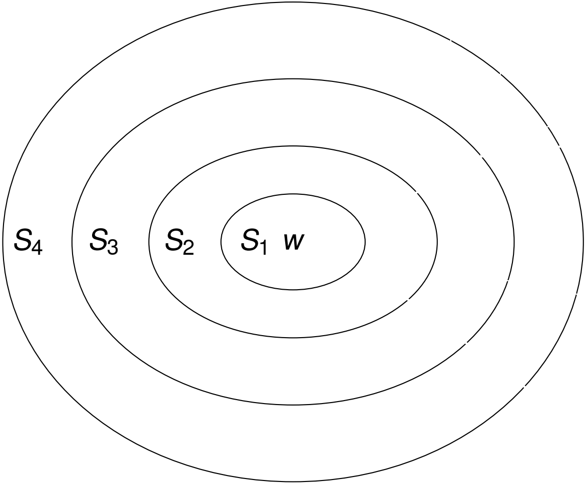
Monotonicity property: if $\varphi \rightarrow \psi$ is valid, then so is $(\varphi \wedge \chi) \rightarrow \psi$.

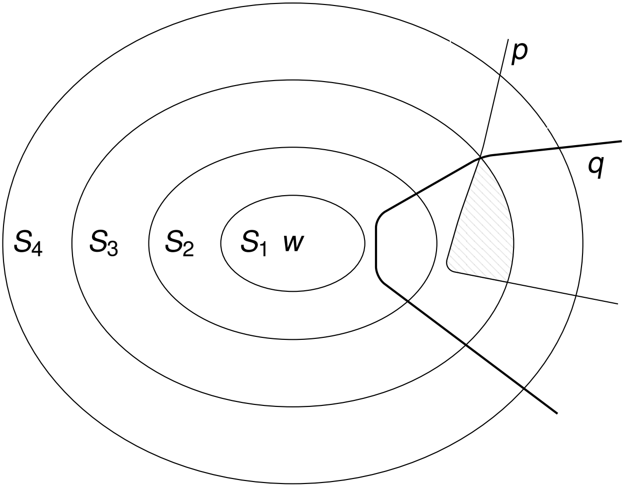
Monotonicity property: if $\varphi \to \psi$ is valid, then so is $(\varphi \wedge \chi) \to \psi$.

Claim. The rule:

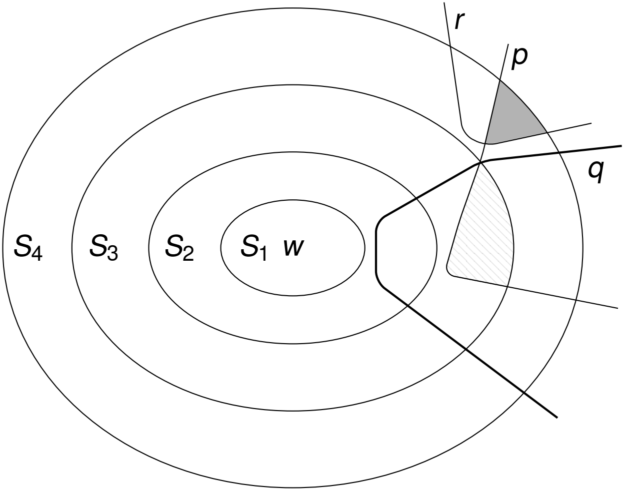
$$\text{if } \varphi \mathrel{\Box\!\!\rightarrow} \psi \text{ is valid, then so is } (\varphi \wedge \chi) \mathrel{\Box\!\!\rightarrow} \psi$$

is not valid.


S_4
S_3
S_2
S_1 w



$$\mathcal{M}, w \models p \;\Box\!\!\rightarrow q$$



$\mathcal{M}, w \models p \mathrel{\Box\!\!\rightarrow} q \qquad \mathcal{M}, w \not\models (p \wedge r) \mathrel{\Box\!\!\rightarrow} q$

Transitivity: $((\varphi \to \psi) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$ is valid

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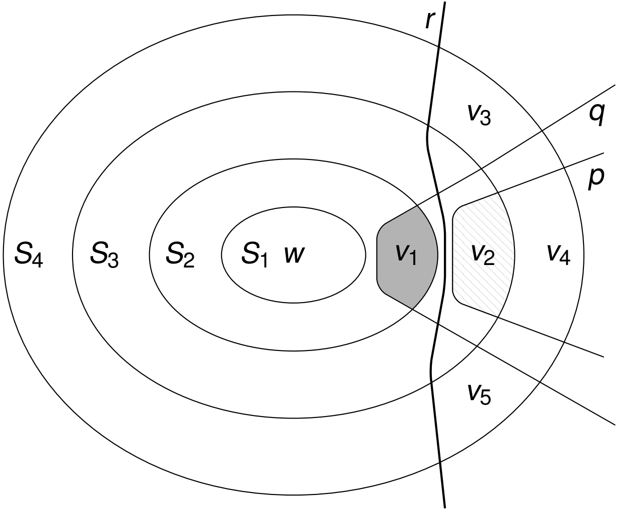
Claim. The axiom:

$$((\varphi \mathrel{\Box\!\!\to} \psi) \wedge (\psi \mathrel{\Box\!\!\to} \chi)) \to (\varphi \mathrel{\Box\!\!\to} \chi)$$

is not valid.



$$\mathcal{M} \models (p \mathrel{\Box\!\!\rightarrow} q) \wedge (q \mathrel{\Box\!\!\rightarrow} r) \wedge \neg(p \mathrel{\Box\!\!\rightarrow} r)$$

Other Modalities

Outer modality

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Inner modality

$\mathcal{M}, w \models [i]\varphi$ iff there is some $S \in N(w)$ such that $\emptyset \neq S \subseteq [\![\varphi]\!]_{\mathcal{M}}$.

$$[i]\varphi \leftrightarrow \top \mathrel{\Box\!\!\rightarrow} \varphi.$$

Other Modalities

Binary Modality

$\varphi \preceq \psi$ means "φ is *at least as possible* as ψ" or "it is no more far-fetched that φ than that ψ"

- $\mathcal{M}, w \models \varphi \preceq \psi$ iff for all $S \in N(w)$, if $S \cap [\![\psi]\!]_{\mathcal{M}} \neq \emptyset$, then $S \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$.
- $\mathcal{M}, w \models \varphi \prec \psi$ iff there is an $S \in N(w)$ such that $S \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and $S \cap [\![\psi]\!]_{\mathcal{M}} = \emptyset$.

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- $\mathcal{M}, w \models \varphi \prec \psi$ iff there is an $S \in N(w)$ such that $S \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and $S \cap [\![\psi]\!]_{\mathcal{M}} = \emptyset$.

$$
\begin{array}{rcl}
(\varphi \mathrel{\Box\!\!\rightarrow} \psi) & \leftrightarrow & ((\varphi \prec \bot) \rightarrow ((\varphi \wedge \psi) \prec (\varphi \wedge \neg\psi))) \\
(\varphi \prec \psi) & \leftrightarrow & \langle o \rangle(\varphi \vee \psi) \wedge ((\varphi \vee \psi) \mathrel{\Box\!\!\rightarrow} \psi)
\end{array}
$$

A (Dynamic) Logic of Knowledge, Evidence and Belief

J. van Benthem and EP. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica, 99, pp. 61 - 92, 2011.

J. van Benthem, D. Fernández-Duque and EP. *Evidence Logic: A New Look at Neighborhood Structures*. Proceedings of Advances in Modal Logic, King's College Publications, 2012.

J. van Benthem, D. Fernández-Duque and EP. *Evidence and Plausibility in Neighborhood Structures*. Annals of Pure and Applied Logic, 2013.

Setting the Stage: Evidence

- Dempster-Shafer Theory of Evidence

G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

- Bayesian Confirmation Theory (eg., E confirms H iff $p(H \mid E) > p(H)$)

B. Fitelson. *The Plurality of Bayesian Measures of Confirmation and the Problem of Measure Sensitivity*. Philosophy of Science 66, 1999.

Setting the Stage: Evidence

- Artemov/Fitting's Justification Logic ($t : \varphi$: "t is a *justification*/*proof* for φ")

S. Artemov and M. Fitting. *Justification logic*. The Stanford Encyclopedia of Philosophy, 2012.

- Moss and Parikh's "topologic" ($x, U \models \varphi$: "φ is true at the state x given that the current *evidence*/*"measurement"* gathered is U")

L. Moss and R. Parikh. *Topological reasoning and the logic of knowledge*. Proceedings of TARK IV, Morgan Kaufmann, 1992.

Setting the Stage: Reasons

- Kratzer Semantics (modal base), believing for a *reason* (deriving an ordering on worlds from an ordering over propositions)

A. Kratzer. *What* must *and* can *must and can mean*. Linguistics and Philosophy 1 (1977) 337355.

C. List and F. Dietrich. *Reasons for (prior) belief in bayesian epistemology*. 2012.

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C. List and F. Dietrich. *Reasons for (prior) belief in bayesian epistemology*. 2012.

- Reason management (Default logic with priorities)

J. Horty. *Reasons as Defaults*. 2012.

Modeling Evidence: Some Distinctions

Barest view: the evidence is encoded as the current range of worlds the agent considers possible

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Ignores how we arrived at this epistemic state

Richest view: complete syntactic details of what we have learned so far (including the sources of each piece of evidence)

In between: family of subsets representing evidence from received from various (possible unreliable) sources

Evidence Models: Basic Assumptions

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2. The evidence gathered from different sources (or even the same source) may be jointly inconsistent. And so, the intersection of all the gathered evidence may be empty.

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2. The evidence gathered from different sources (or even the same source) may be jointly inconsistent. And so, the intersection of all the gathered evidence may be empty.

3. Despite the fact that sources may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.

Evidential States

An **evidential state** is a collection of subsets of W.

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Assumptions:

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- The whole universe W is an evidence set (agents know their 'space').

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In addition, much of the literature would suggest a 'monotonicity' assumption:

> *If the agent has evidence X and $X \subseteq Y$ then the agent has evidence Y.*

Example: $W = \{w, v\}$ where p is true only at w

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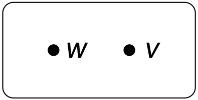
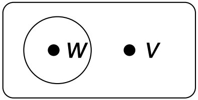
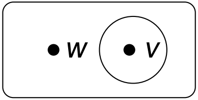


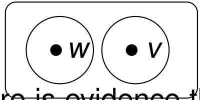
There is no evidence for or against p.



There is evidence that supports p.



There is evidence that rejects p.



There is evidence that supports p and also evidence that rejects p.

Evidence Models

Evidence model: $\mathcal{M} = \langle W, E, V\rangle$

- W is a non-empty set of worlds,
- $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function, and
- $E : W \rightarrow \wp(\wp(W))$ is an evidence relation

  $X \in E(w)$: "the agent accepts X as evidence at state w".

Uniform evidence model (E is a constant function):
$\langle W, \mathcal{E}, V\rangle, w$ where $\mathcal{E}$ is the fixed family of subsets of W related to each state by E.

Assumptions

(Cons) For each state w, $\emptyset \notin E(w)$.

(Triv) For each state w, $W \in E(w)$.

The Basic Language $\mathcal{L}$ of Evidence and Belief

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\,]\varphi \mid [B]\varphi \mid [A]\varphi$$

- $\langle\,]\varphi$ says that "the agent has evidence that φ is true" (i.e., "the agent has evidence for φ")
- $[B]\varphi$ says that "the agents believes that φ is true" (based on her evidence)
- $[A]\varphi$ says that "φ is true in all states" (which we interpret as the agent's *knowledge*)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \mathsf{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

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- $\mathcal{M}, w \models \langle\,]\varphi$ iff there exists X such that $X \in E(w)$ and for all $v \in X$, $\mathcal{M}, v \models \varphi$

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- $\mathcal{M}, w \models \langle\,]\varphi$ iff there exists X such that $X \in E(w)$ and for all $v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

“Having evidence for φ” vs. “Accepting φ as evidence”

We do not assume that the evidence sets are closed under supersets, though our semantic definition implies that the set of propositions that the agent has *evidence for* is closed under weakening.

So, an agent can have *evidence for* X without *accepting* the set X as evidence.

Defining Beliefs

***w*-scenario**: A maximal family of evidence sets $\mathcal{X} \subseteq E(w)$ that has the **finite intersection property** (f.i.p.: for each finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{X}$, $\bigcap_{1 \leq i \leq n} X_i \neq \emptyset$).

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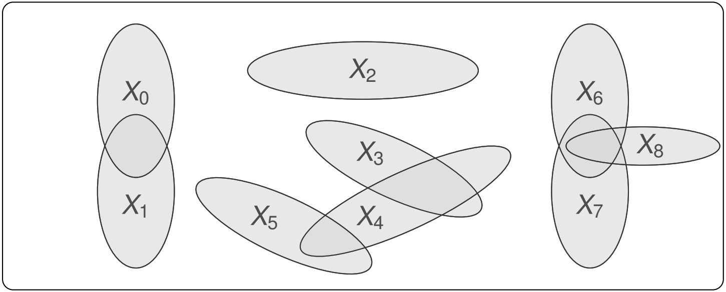
An agent believes φ at w if each w-scenario implies that φ is true (i.e., φ is true at each point in the intersection of each w-scenario).

Defining Beliefs



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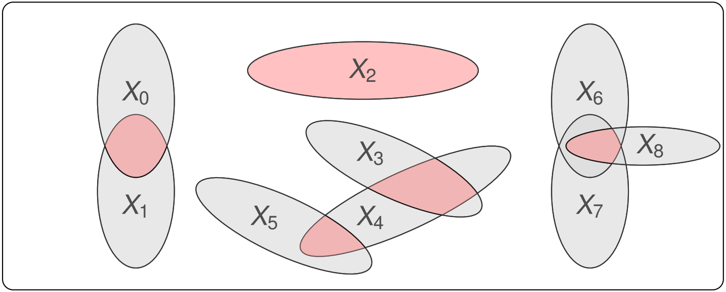

Our definition of belief is very conservative, many other definitions are possible (there exists a w-scenario, "most" of the w-scenarios,...)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \mathrm{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there exists X such that wEX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
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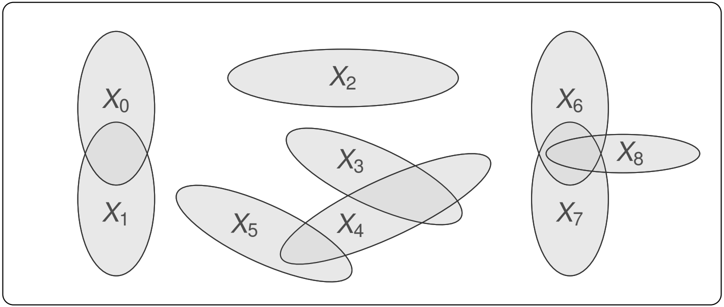
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [B]\varphi$ for all w-scenarios $\mathcal{X} \subseteq E(w)$, for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$

Notation for the truth set: $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$

Conditional Beliefs on Evidence Models



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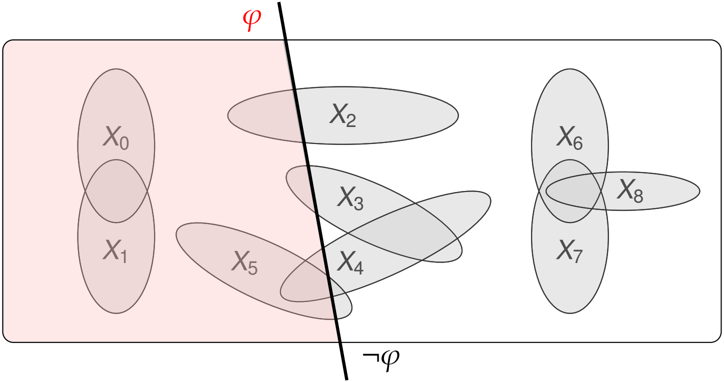

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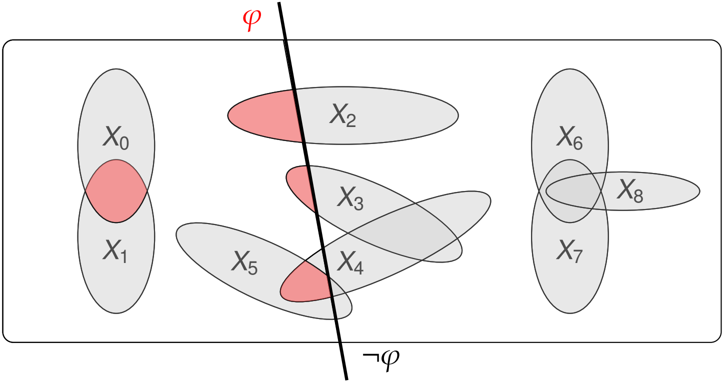

Conditional Beliefs on Evidence Models

$B^{\varphi}\psi$: "the agent believes ψ conditional on φ."

Main idea: Ignore the evidence that is inconsistent with φ.

Relativized *w*-scenario: Suppose that $X \subseteq W$. Given a collection $\mathcal{X} \subseteq \wp(W)$, let $\mathcal{X}^X = \{Y \cap X \mid Y \in \mathcal{X}\}$. We say that a collection $\mathcal{X}$ of subsets of W has the **finite intersection property relative to** X (X-f.i.p.) if, $\mathcal{X}^X$ as the f.i.p. and is maximal if $\mathcal{X}^X$ is.

- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

Conditional Beliefs: Example

$B\psi \to B^{\varphi}\psi$ is not valid.

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Is $B\psi \to B^{\varphi}\psi \vee B^{\neg\varphi}\psi$ valid?

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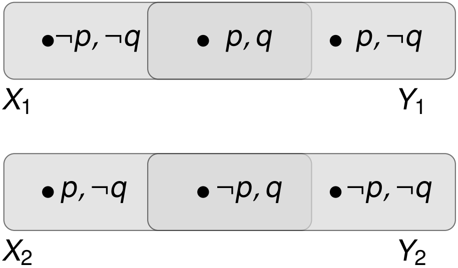

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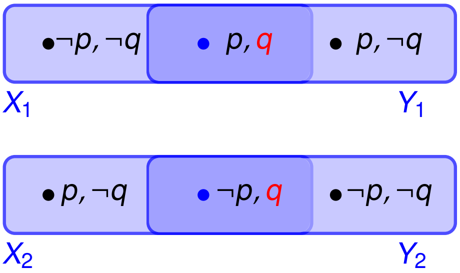

- $\mathcal{M}, w \models Bq$

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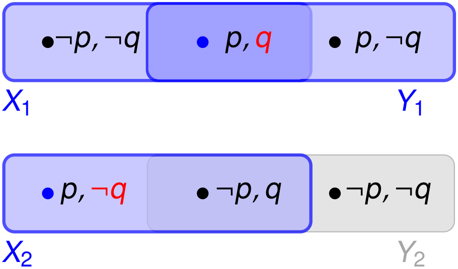

- ✓ $\mathcal{M}, w \models Bq$
- $\mathcal{M}, w \not\models B^{p}q$

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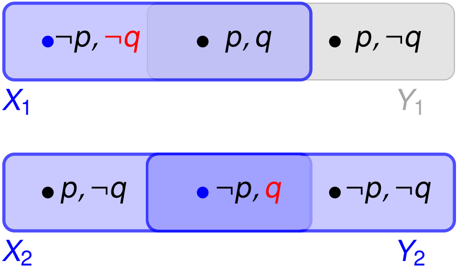
$B\psi \to B^{\varphi}\psi$ is not valid.

Is $B\psi \to B^{\varphi}\psi \vee B^{\neg\varphi}\psi$ valid? No



- ✓ $\mathcal{M}, w \models Bq$
- ✓ $\mathcal{M}, w \not\models B^{p}q$
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Conditional Evidence

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$\langle\,]^{\varphi}\psi$ is not equivalent to $\langle\,](\varphi \to \psi)$: if there is no evidence consistent with φ, then $\langle\,]^{\varphi}\psi$ is false.

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \mathsf{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \langle\,]\varphi$ iff there exists X such that wEX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
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- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [B]\varphi$ for all w-scenarios $\mathcal{X} \subseteq E(w)$, for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

Flat Evidence Models

An evidence model $\mathcal{M}$ is **flat** if every scenario on $\mathcal{M}$ has non-empty intersection.

Proposition. The formula $\langle\ ]\varphi \to \langle B\rangle\varphi$ is valid on the class of flat evidence models, but not on the class of all evidence models.

1. Prove that $\langle\ ]\varphi \wedge [A]\psi \leftrightarrow \langle\ ](\varphi \wedge [A]\psi)$ is valid on all evidence models.

2. Prove that $[B]\varphi \rightarrow [A][B]\varphi$ is valid on all uniform evidence models.

3. Show that $\langle\ ]\varphi \rightarrow \langle\ ]\langle\ ]\varphi$ is only valid on uniform evidence models.

- ✓ Subset spaces, neighborhood frames/models, reasoning about subset spaces
- ✓ Logic of knowledge, evidence and belief
- ▸ Coalitional logic

Coalitional Logic

M. Pauly. *A Modal Logic for Coalitional Powers in Games*. Journal of Logic and Computation, 12:1, pp. 149 - 166, 2002.

M. Pauly. *Logic for Social Software*. PhD Thesis, Institute for Logic, Language and Computation, 2001.

Strategic Game Forms

$$\langle N, \{S_i\}_{i \in N}, O, o \rangle$$

- N is a finite set of players;
- for each $i \in N$, S_i is a non-empty set (elements of which are called actions or strategies);
- O is a non-empty set (elements of which are called **outcomes**); and
- $o : \Pi_{i \in N} S_i \to O$ is a function assigning an outcome

		Bob	
		t_1	t_2
Ann	s_1	o_1	o_2
	s_2	o_2	o_3
	s_3	o_4	o_1

α-Effectivity

$S = \Pi_{i \in N} S_i$ are called **strategy profiles**. Given a strategy profile $s \in S$, let s_i denote i's component and s_{-i} the profile of strategies from s for all players except i.

A strategy for a coalition C is a sequence of strategies for each player in C, i.e., $s_C \in \Pi_{i \in C} S_i$ (similarly for $s_{\overline{C}}$, where $\overline{C}$ is $N - C$).

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Suppose that $G = \langle N, \{S_i\}_{i \in N}, O, o \rangle$ be a strategic game form. An α**-effectivity function** is a map $E_G^\alpha : \wp(N) \rightarrow \wp(\wp(O))$ defined as follows: For all $C \subseteq N$, $X \in E_G^\alpha(C)$ iff there exists a strategy profile s_C such that for all $s_{\overline{C}} \in \Pi_{i \in N-C} S_i$, $o(s_C, s_{\overline{C}}) \in X$.

α-Effectivity vs. β-Effectivity

$\exists$ "something a player/a coalition *can* do" such that $\forall$ "actions of the other players/nature"...

α-Effectivity vs. β-Effectivity

$\exists$ "something a player/a coalition *can* do" such that $\forall$ "actions of the other players/nature"...

$\forall$ "(joint) actions of the other players", $\exists$ "something the agent/coalition can do"...

		Bob	
		t_1	t_2
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$$E^{\alpha}_{G_0}(\{A\}) = sup(\{\{o_1, o_2\}, \{o_2, o_3\}, \{o_1, o_4\}\})$$

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$$E^{\alpha}_{G_0}(\{A, B\}) = sup(\{o_1\}, \{o_2\}, \{o_3\}, \{o_4\}\}) = \wp(O) - \emptyset$$

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$$E^{\alpha}_{G_0}(\emptyset) = \{\{o_1, o_2, o_3, o_4, o_5, o_6\}\}$$

Playable Effectivity Functions

1. (*Liveness*) For all $C \subseteq N$, $\emptyset \notin E(C)$

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$E(\{i\}) = \{X \mid X \subseteq \mathbb{N} \text{ is infinite}\}$;

$E(\emptyset) = \{X \mid X \subseteq \mathbb{N} \text{ is cofinite (i.e., } \overline{X} \text{ is finite)}\}$;

$$E(\{i\}) = \{X \mid X \subseteq \mathbb{N} \text{ is infinite}\};$$
$$E(\emptyset) = \{X \mid X \subseteq \mathbb{N} \text{ is cofinite (i.e., } \overline{X} \text{ is finite)}\};$$

Claim. E satisfies Liveness, Safety, N-maximality, Outcome Monotonicity, Superadditivity, but is not the effectivity function of any game.

Core-Complete

Suppose that $(W, \mathcal{F})$ is a monotonic subset space. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F} \text{ and for all } X' \subseteq W, \text{ if } X' \subsetneq X, \text{ then } X' \notin \mathcal{F}\}.$$

Does every subset space $(W, \mathcal{F})$ have a non-monotonic core?

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Does every subset space $(W, \mathcal{F})$ have a non-monotonic core? No.

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

Observation. Suppose that $G = \langle N, \{S_i\}_{i \in N}, O, o \rangle$ is a strategic game form and E_G^α is the associated α-effectivity function. Then the non-monotonic core of $E_G^\alpha(\emptyset) = \{range(o)\}$, where $range(o) = \{x \in O \mid \text{there is a } s \in \Pi_{i \in N} S_i \text{ such that } o(s) = x\}$.

Observation. Suppose that $G = \langle N, \{S_i\}_{i \in N}, O, o\rangle$ is a strategic game form and E^{α}_{G} is the associated α-effectivity function. Then the non-monotonic core of $E^{\alpha}_{G}(\emptyset) = \{range(o)\}$, where $range(o) = \{x \in O \mid \text{there is a } s \in \Pi_{i \in N} S_i \text{ such that } o(s) = x\}$.

Claim. If $E(\emptyset) = \{Y \mid Y \text{ is co-finite}\}$, then $E^{nc}(\emptyset) = \emptyset$.

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6. (*Empty Coalition*) $E(\emptyset)$ is core complete.

Characterizing Playable Effectivity Functions

Theorem (Pauly 2001; Goranko, Jamorga and Turrini 2013). If $E : \wp(N) \rightarrow \wp(\wp(O))$ is a function that satisfies the conditions 1-6 given above, then $E = E^{\alpha}_{G}$ for some strategic game form.

V. Goranko, W. Jamroga, and P. Turrini. *Strategic Games and Truly Playable Effectivity Functions*. Journal of Autonomous Agents and Multiagent Systems, 26(2), pgs. 288 - 314, 2013.

M. Pauly. *Logic for Social Software*. PhD Thesis, Institute for Logic, Language and Computation, 2001.

Coalitional Models

A coalitional logic model is a tuple $\mathcal{M} = \langle W, E, V \rangle$ where W is a set of states, $E : W \to (\wp(N) \to \wp(\wp(W)))$ assigns to each state a playable effectivity function, and $V : \mathsf{At} \to \wp(W)$ is a valuation function.

$$\mathcal{M}, w \models [C]\varphi \text{ iff } [\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\} \in E(w)(C)$$

Coalitional Logic: Axiomatics

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3. (*N-maximality*) For all $X \subseteq O$, if $X \in E(N)$ then $\overline{X} \notin E(\emptyset)$

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5. (*Superadditivity*) $([C_1]\varphi_1 \wedge [C_2]\varphi_2) \rightarrow [C_1 \cup C_2](\varphi_1 \wedge \varphi_2)$, where $C_1 \cap C_2 = \emptyset$

The Broader Logical Landscape

- Relational Models
- Topological Models
- *n*-ary Relational Structures
- Plausibility Structures
- First-Order Logic

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

From Kripke Frames to Neighborhood Frames

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Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is *R-necessary at w* if $R^{\rightarrow}(w) \subseteq X$.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W\text{, let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

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Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}_w^R$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$
- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

Theorem

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From Neighborhood Frames to Kripke Frames

Theorem

✓ *Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. $\square$

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

✓ *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.
For each $w, v \in W$, wR_Nv iff $v \in \cap N(w)$. □

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- Bisimulation
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- **EC** $\nvdash K$
- **EM** $\nvdash K$
- **EMCN** $\vdash K$
- **EK** $\nvdash M$
- **EK** $\nvdash C$

Expressive Power and Invariance

M. Pauly. *Bisimulation for Non-normal Modal Logic*. Manuscript, 1999.

H. Hansen. *Monotonic Modal Logic*. ILLC, Masters Thesis, 2003.

Monotonic Bisimulation

Suppose that $\mathfrak{M} = \langle W, N, V\rangle$ and $\mathfrak{M}' = \langle W', N', V'\rangle$ are two monotonic neighborhood models. A relation $Z \subseteq W \times W'$ is a **monotonic bisimulation** provided that, whenever wZw':

Atomic harmony: for each $p \in \mathrm{At}$, $w \in V(p)$ iff $w' \in V'(p)$.

Zig: If $w\ N\ X$ then there is an $X' \subseteq W'$ such that $w'\ N'\ X'$ and $\forall x' \in X', \exists x \in X$ such that $x\ Z\ x'$.

Zag: If $w'\ N'\ X'$ then there is an $X \subseteq W$ such that $w\ N\ X$ and $\forall x \in X, \exists x' \in X'$ such that $x\ Z\ x'$.

Write $\mathfrak{M}, w \,\underline{\leftrightarrow}\, \mathfrak{M}', w'$ when there is a monotonic bisimulation $Z \subseteq dom(\mathcal{M}) \times dom(\mathcal{M}')$ such that $w\ Z\ w'$.

Proposition. If $\mathcal{M}$ is a monotonic model, $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ implies $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$.

Locally Core-Finite Models

Suppose that $\mathcal{F}$ is a monotonic collection of subsets of W. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F} \text{ and for all } X' \subseteq W, \text{ if } X' \subseteq X, \text{ then } X' \notin \mathcal{F}\}.$$

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

Locally Core-Finite Models

Suppose that $\mathcal{F}$ is a monotonic collection of subsets of W. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F} \text{ and for all } X' \subseteq W, \text{ if } X' \subsetneq X, \text{ then } X' \notin \mathcal{F}\}.$$

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

Question: Is every monotonic collection core-complete?

Locally Core-Finite Models

A neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ is **locally core-finite** provided that $\mathcal{M}$ is core-complete and for each $w \in W$, $N^{nc}(w)$ is finite, and for all $X \in N^{nc}(w)$, X is finite.

Proposition. Suppose that $\mathcal{M} = \langle W, N, V\rangle$ and $\mathcal{M}' = \langle W', N', V'\rangle$ are monotonic, locally core-finite models. Then, for all $w \in W$, $w' \in W'$, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Do monotonic bisimulations work when we drop monotonicity?
No!

$$
\begin{tikzcd}
 & \{w_1\} & & \{v_1\} \\
w_2 & w_1 \arrow[u] \arrow[rr, dashed, dash, bend left, "Z"] & & v_1 \arrow[u]
\end{tikzcd}
$$

$$\mathcal{M} \qquad\qquad\qquad \mathcal{M}'$$

$$V(p) = \{w_1, w_2\} \qquad\qquad V'(p) = \{v_1\}$$

Bounded Morphisms

If $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models, and $f : W_1 \to W_2$ is a function, then f is a **(frame) bounded morphism** if

for all $X \subseteq W_2$, we have $f^{-1}[X] \in N_1(w)$ iff $X \in N_2(f(w))$;

and for all $p \in \text{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

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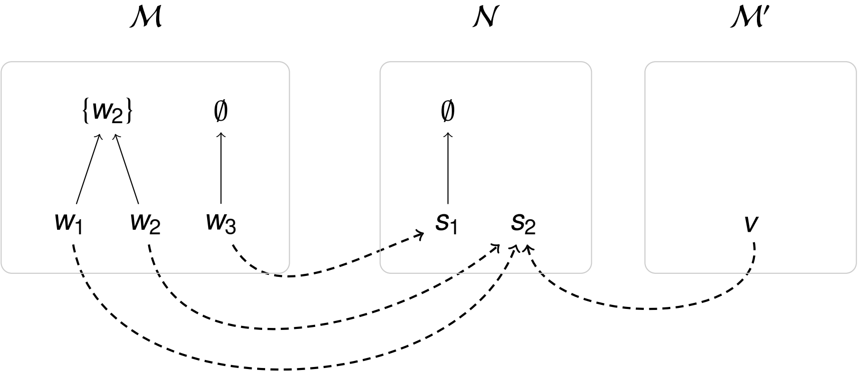
Lemma Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models and $f : \mathcal{M}_1 \to \mathcal{M}_2$ a bounded morphism. For each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathcal{M}_1, w \models \varphi$ iff $\mathcal{M}_2, f(w) \models \varphi$.

Behavioral Equivalence

Definition

Two points w_1 from $\mathfrak{M}_1$ and w_2 from $\mathfrak{M}_2$ are **behaviorally equivalent** provided there is a neighborhood frame $\mathfrak{F}$ and bounded morphisms $f : \mathfrak{F}_1 \to \mathfrak{F}$ and $g : \mathfrak{F}_2 \to \mathfrak{F}$ such that $f(w_1) = g(w_2)$.


$\mathcal{M}$
$\mathcal{N}$
$\mathcal{M}'$
$\{w_2\}$
$\emptyset$
$\emptyset$
w_1
w_2
w_3
s_1
s_2
v

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are two neighborhood models. If states $w \in W$ and $w' \in W'$ are behaviorally equivalent, then for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

Disjoint Union

Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models. The **disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$** is the neighborhood model $\mathcal{M}_1 + \mathcal{M}_2 = \langle W_1 + W_2, N, V \rangle$ where for all $p \in \mathrm{At}$, $V(p) = V_1(p) \cup V_2(p)$; and for $i = 1, 2$,

for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

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for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

Proposition. For all $\varphi \in \mathcal{L}$, for $i = 1, 2$, if $w \in W_i$, then $\mathcal{M}_1 + \mathcal{M}_2, w \models \varphi$ iff $\mathcal{M}_i, w \models \varphi$.

Fact. The universal modality is not definable in the basic modal language.

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- ✓ Bisimulations
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Suppose that Γ is a set of formulas and F is a class of neighborhood frames. A formula $\varphi \in \mathcal{L}$ is a **semantic consequence** with respect to F of Γ, denoted $\Gamma \models_{\mathsf{F}} \varphi$, provided for each model $\mathcal{M} = \langle W, N, V\rangle$ based on a frame from F (i.e., $\langle W, N\rangle \in \mathsf{F}$), for each $w \in W$, if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$.

Some Notation

- A formula $\varphi \in \mathcal{L}$ is valid in F ($\models_\mathsf{F} \varphi$) if for each $\mathbb{F} \in$ F, $\mathbb{F} \models \varphi$.

- We say that a logic **L** is sound with respect to F, provided $\vdash_\mathbf{L} \varphi$ implies $\models_\mathsf{F} \varphi$.

- A logic **L** is weakly complete with respect to a class of frames F, if $\models_\mathsf{F} \varphi$ implies $\vdash_\mathbf{L} \varphi$.

- A logic **L** is strongly complete with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_\mathsf{F} \varphi$ implies $\Gamma \vdash_\mathbf{L} \varphi$.

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A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas.

The **L**-proof set of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Let **L** be a logic and $\varphi, \psi \in \mathcal{L}$. Then

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. For any maximally **L**-consistent set Γ, if $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$
7. For any maximally **L**-consistent set Γ, If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$

Lindenbaum's Lemma. For any consistent set of formulas Γ, there exists a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

Canonical Model

Definition

A neighborhood model $\mathbb{M} = \langle W, N, V\rangle$ is canonical for **L** provided

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A neighborhood model $\mathbb{M} = \langle W, N, V\rangle$ is **canonical for $\mathbf{L}$** provided

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- for all $p \in \mathrm{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathcal{M}^{min}_{\mathbf{L}} = \langle M_{\mathbf{L}}, N^{min}_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$, $N^{min}_{\mathbf{L}}(\Gamma) = \{|\varphi|_{\mathbf{L}} \mid \Box\varphi \in \Gamma\}$.

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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathcal{M}^{max}_{\mathbf{L}} = \langle M_{\mathbf{L}}, N^{max}_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N^{max}_{\mathbf{L}}(\Gamma) = N^{min}_{\mathbf{L}}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin P_{\mathbf{L}}\}$

The canonical model works...

Lemma

For any logic $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Lemma (Truth Lemma)

For any consistent classical modal logic $\mathbf{L}$ *and any consistent formula* φ, *if* $\mathcal{M}$ *is canonical for* $\mathbf{L}$,

$$[\![\varphi]\!]_{\mathcal{M}} = |\varphi|_{\mathbf{L}}$$

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The Proofs

Theorem

The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

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Lemma

If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem

The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

The Proofs

Fact: $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}}\rangle$ is not closed under supersets.

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Fact: $\langle M_{\mathbf{EM}}, N_{\mathbf{EM}}^{min}\rangle$ is not closed under supersets.

Lemma

Suppose that $\mathcal{M} = sup(\mathcal{M}_{\mathbf{EM}}^{min})$. Then $\mathcal{M}$ is canonical for **EM**.

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

The Proofs

Theorem

The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem

The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

The Normal Situation

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \rightarrow \psi \qquad \vdash \varphi}{\psi}$

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Theorem: K is sound and strongly complete with respect to the class of all Kripke frames.

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Theorem: For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

The Normal Situation

The smallest normal modal logic **K** consists of

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MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

Theorem: **K** $+ \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive Kripke frames.

A logic **L** is **neighborhood complete** (resp. **Kripke complete**) provided there is a class of neighborhood frames F (resp. relational frames) such that
$\mathbf{L} = \mathbf{L}(\mathsf{F}) = \{\varphi \in \mathcal{L} \mid \mathbb{F} \models \varphi \text{ for all } \mathbb{F} \in \mathsf{F}\}$. Otherwise, the logic is said to be **neighborhood incomplete** (resp. **Kripke incomplete**).

Incompleteness

There are (consistent) modal logics that are incomplete:

Incompleteness

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Theorem Let **TMEQ** be the following normal modal logic:

- **K**
- $\Box\varphi \rightarrow \varphi$
- $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$
- $\Diamond(\Diamond\varphi \wedge \Box\psi) \rightarrow \Box(\Diamond\varphi \vee \Box\varphi)$
- $(\Diamond\varphi \wedge \Box(\varphi \rightarrow \Box\varphi)) \rightarrow \varphi$

There is no class of frames validating precisely the formulas in **TMEQ**.

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There is no class of frames validating precisely the formulas in **TMEQ**.

J. van Benthem. *Two Simple Incomplete Modal Logics*. Theoria (1978).

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

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There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for **L** and **L**$'$ respectively, but φ and φ' are not deducible in the respective logics).

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**′ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L′ is above **S4** (adapts Fine's incomplete logic)

$$
\begin{aligned}
A_i &= \Box(q_i \to r) \qquad (i = 1,2)\\
B_i &= \Box(r \to \Diamond q_i) \qquad (i = 1,2)\\
C_1 &= \Box\neg(q_1 \wedge q_2)\\
A &= r \wedge \Box p \wedge \neg\Box\Box p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge\\
&\quad C_1 \to \Diamond(r \wedge \Box(r \to (q_1 \vee q_2))\\
D &= (p \wedge \Diamond\Diamond q) \to (\Diamond q \vee \Diamond\Diamond(q \wedge \Diamond p))\\
E &= (\Box p \wedge \neg\Box\Box p) \to \Diamond(\Box\Box p \wedge \neg\Box\Box\Box p)\\
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Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

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Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

Theorem. (Gerson) The formula F is valid in all neighborhood frames for **L**, but it is not provable in **L**.

Comparing Relational and Neighborhood Semantics

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Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

Neighborhood completeness does not imply Kripke completeness

- extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

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L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

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T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

Recovering Completeness

Definition

A general neighborhood frame is a tuple $\mathfrak{F}^g = \langle W, N, \mathcal{A}\rangle$, where $\langle W, N\rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

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A valuation $V : \text{At} \to \wp(W)$ is admissible for a general frame $\langle W, N, \mathcal{A} \rangle$ if for each $p \in \text{At}$, $V(p) \in \mathcal{A}$.

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Definition

Suppose that $\mathfrak{F}^g = \langle W, N, \mathcal{A} \rangle$ is a general neighborhood frame. A general modal based on $\mathfrak{F}^g$ is a tuple $\mathfrak{M}^g = \langle W, N, \mathcal{A}, V \rangle$ where V is an admissible valuation.

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Lemma

Let $\mathfrak{M}^g$ be an general neighborhood model. Then for each $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathfrak{M}^g} \in \mathcal{A}$.

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Definition

A general neighborhood frame is a tuple $\mathfrak{F}^g = \langle W, N, \mathcal{A} \rangle$, where $\langle W, N \rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

Lemma

Let **L** *be any logic extending* **E***. Then the general canonical frame validates* **L** *(*$\mathfrak{F}^g_{\mathbf{L}} \models \mathbf{L}$*).*

Corollary

Any modal logic extending **E** *is strongly complete with respect to some class of general frames.*

Summary

For any modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of general frames

Summary

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There are modal logics showing that

- neighborhood completeness does not imply Kripke completeness
- algebraic completeness does not imply neighborhood completeness