CMSC427 Geometry and Vectors: Metrics and the Dot Product

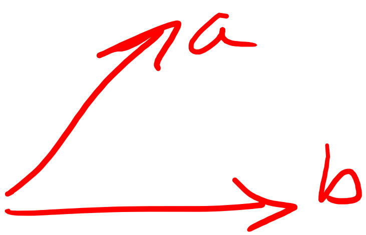
Euclidean geometry

Add distance metric to Affine space

Measure distances and angles

Inner product

dot product $\underline{2D, 3D,}$ ND



$$a \cdot b = a_x b_x + a_y b_y$$

$$a = \langle a_x, a_y \rangle$$

$$b = \langle b_x, b_y \rangle$$

maps two vectors → scalar

Vector magnitude

The magnitude (length) of a vector is

$$|\mathbf{v}|^2 = v_x{}^2 + v_y{}^2$$

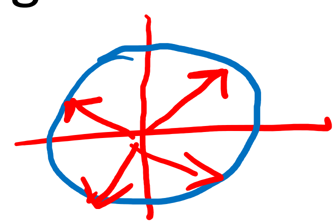
$|v|^2 = v \cdot v$

$$|\mathbf{v}| = \sqrt{v_x{}^2 + v_y{}^2} = \sqrt{v \cdot v}$$

A vector of length 1.0 is called a *unit vector*

To *normalize* a vector is to rescale it to unit length

$$\mathrm{n}_v = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{|v|}{\sqrt{v \cdot v}}$$



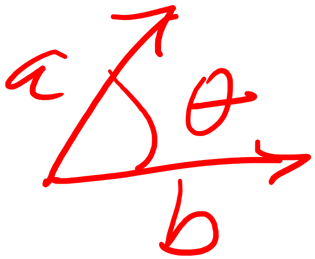
Normal vectors represent direction and are used for:
light direction, surface normals, rotation axes

angle/direction

Dot product

- Inner product between two vectors
- Defined using coordinate-free cosine rule

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$



$\theta = 90^\circ \Rightarrow \cos(\theta) = 0$

$\theta = 0^\circ \Rightarrow \cos(\theta) = 1$

- Example:

$$\mathbf{a} \;=< 1{,}1 > \qquad |a| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\mathbf{b} \;=< 1{,}0 > \qquad |b| = \sqrt{1^2+0^2} = 1$$

$$\theta = \frac{\pi}{4} \qquad \cos(\pi/4) = 1/\sqrt{2}$$

$$\mathbf{a} \bullet \mathbf{b} = 1.4142 * 1 * 0.70711 = 1$$

$$\sqrt{2} * 1 * 1/\sqrt{2} = 1$$

Dot product – coordinate version

- Given

$$\mathbf{a} = \langle \boldsymbol{a_x}, \boldsymbol{a_y} \rangle \qquad \mathbf{b} = \langle \boldsymbol{b_x}, \boldsymbol{b_y} \rangle$$

- Then

$$\mathbf{a} \bullet \mathbf{b} = \boldsymbol{a_x b_x} + \boldsymbol{a_y b_y}$$

- In n dimensions

$a_x b_x + a_y b_y + a_z b_z$

$$\mathbf{a} \bullet \mathbf{b} = \sum_{i=1}^{n} \boldsymbol{a_i b_i}$$

$n > 3$

Dot product – coordinate version

- Given

$$\mathbf{a} = \langle \boldsymbol{a_x}, \boldsymbol{a_y} \rangle \qquad \mathbf{b} = \langle \boldsymbol{b_x}, \boldsymbol{b_y} \rangle$$

- Then

$$\mathbf{a} \bullet \mathbf{b} = \boldsymbol{a_x b_x} + \boldsymbol{a_y b_y}$$

- Example revised:

$$\begin{aligned} \mathbf{a} &=< 1,1 > \\ \mathbf{b} &=< 1,0 > \\ \mathbf{a} \bullet \mathbf{b} = 1 * 1 &+ 1 * 0 = 1 \end{aligned}$$

Dot product: computing angle (and cosine)

- Given $$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$
- We have

$$\cos(\theta) = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

- Example

$$\mathbf{a} = <1,1>, \mathbf{b} = <1,0>$$

$$\mathbf{a} \bullet \mathbf{b} = 1 * 1 + 1 * 0 = 1$$

$$|\mathbf{a}| = \mathbf{1.4142}, |\mathbf{b}| = \mathbf{1.0}$$

$$\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{1}}{\mathbf{1.4142}} = \mathbf{0.70711}$$

$$\cos^{-1}\left(\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \mathbf{0.78540} = \boldsymbol{\pi}/\mathbf{4}$$

$$\frac{a \cdot b}{|a||b|} = \frac{1}{\sqrt{2}(1)} = \frac{1}{\sqrt{2}}$$

$$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$$

Dot product: testing perpendicularity

- Since

$$\cos(90°) = \cos\left(\frac{\pi}{2}\right) = \mathbf{0}$$

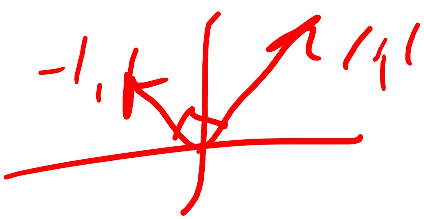
- Then a perpendicular to b gives

$$\mathbf{a} \perp \mathbf{b} \Rightarrow \mathbf{a} \bullet \mathbf{b} = 0$$

- Examples

$$\mathbf{a} = < 1{,}0 >, \mathbf{b} = < 0{,}1 >$$
$$\mathbf{a} \bullet \mathbf{b} = 1 * 0 + 1 * 0 = 0$$

$$\mathbf{a} = < 1{,}1 >, \mathbf{b} = < -1{,}1 >$$
$$\mathbf{a} \bullet \mathbf{b} = 1 * -1 + 1 * 1 = 0$$



Perp vector and lines

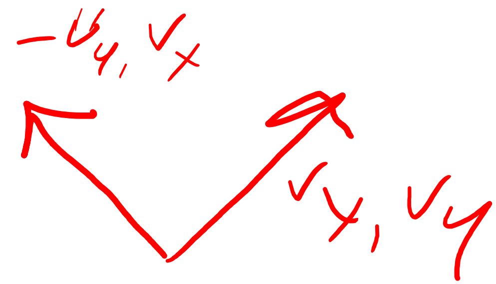
for 2D only

- Given vector

$$\mathbf{v} \; =< v_x, v_y >$$

- The perp vector is

$$=< -v_y, v_x >$$



- So

$$\mathbf{v} \bullet \mathbf{v}^{\perp} = 0$$

$v^{\perp}$

What you should know after today

1. Vector magnitude and normalization
2. Dot products and cosine rule
3. Dot product, perpendicularity and perp vector

also collinearity

$\rightrightarrows$