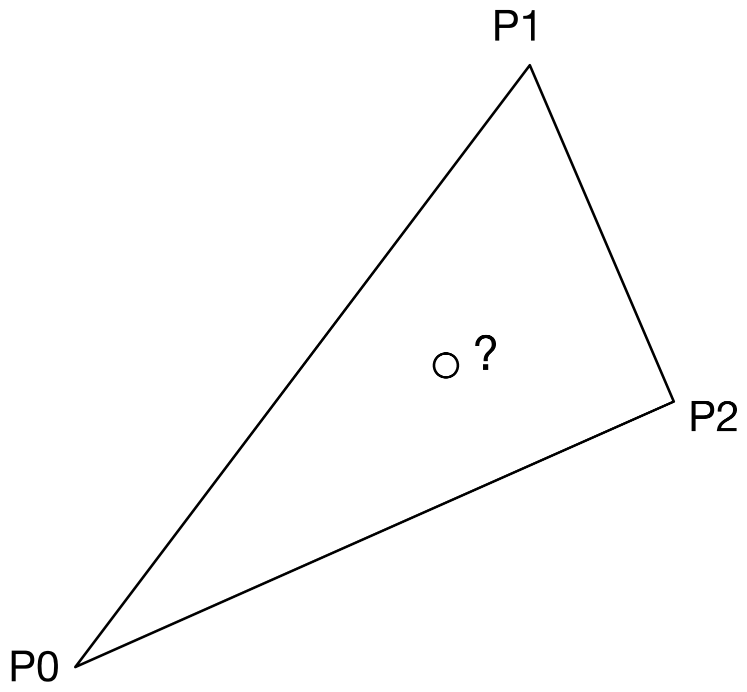
CMSC427 Geometry and Vectors: Affine and Convex Combinations

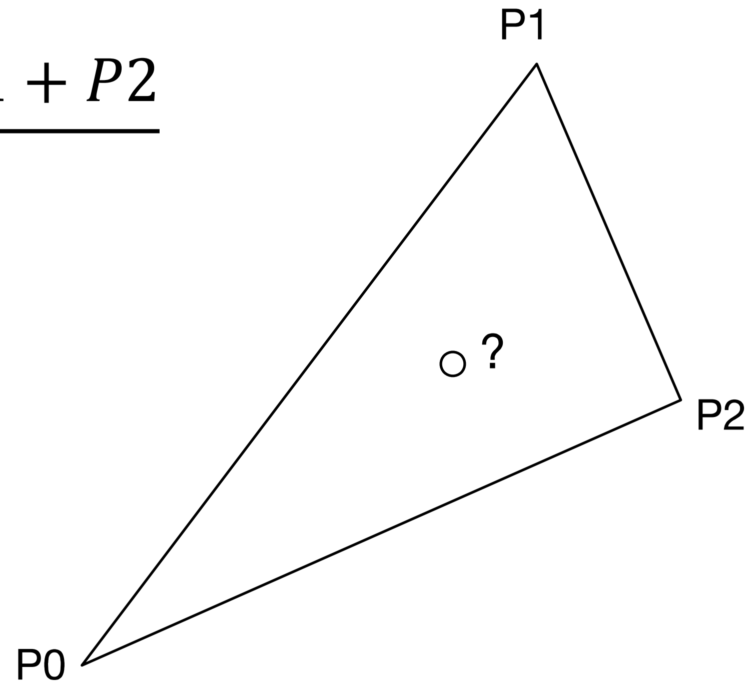
Midpoint of triangle?



Midpoint of triangle?

Answer 1:

$$m = \frac{P0 + P1 + P2}{3}$$



Midpoint of triangle?

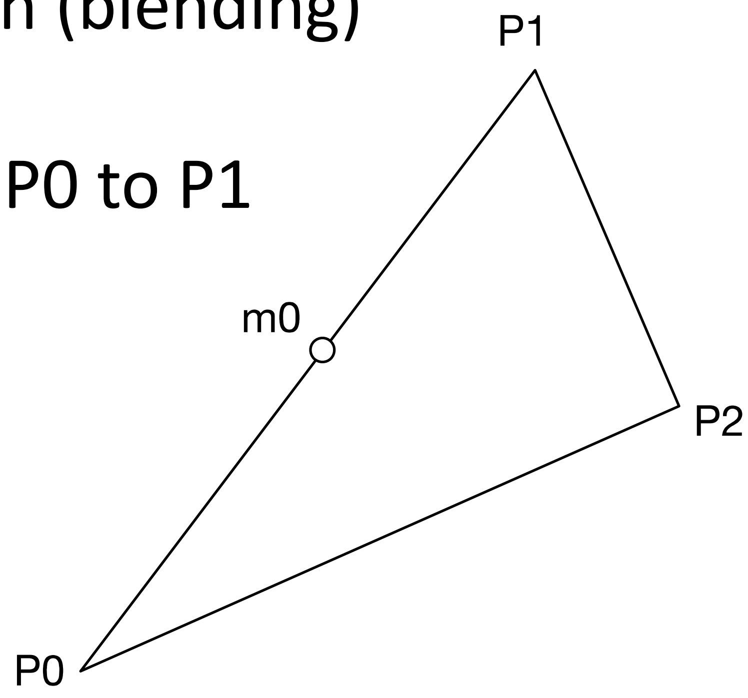
Answer 2: Double interpolation (blending)

First interpolation: vector line P0 to P1

$$P = tV + P0$$
$$= t(P1 - P0) + P0$$
$$= tP1 + (1 - t)P0$$

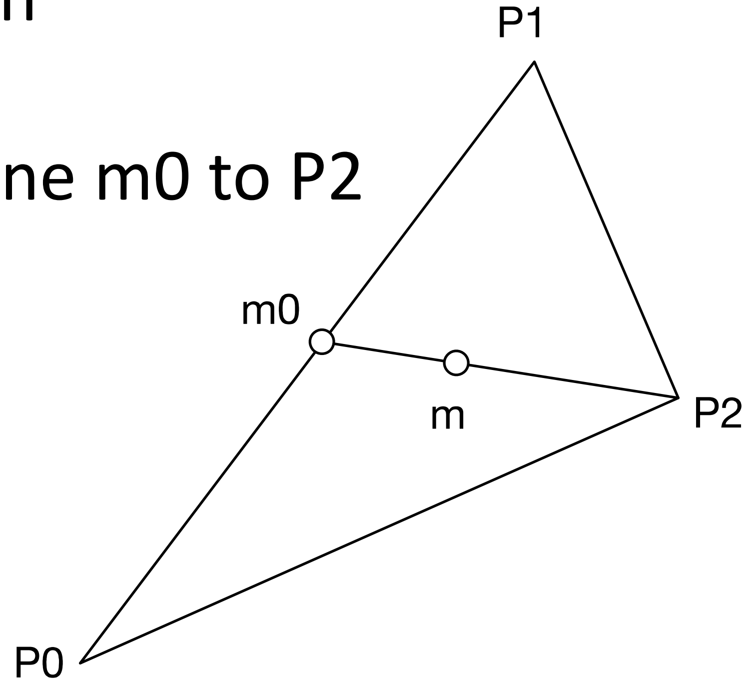
Midpoint at t = 0.5

$$m0 = 0.5P1 + (1 - 0.5)P0$$
$$= 0.5P1 + 0.5P0$$
$$= \frac{P1 + P0}{2}$$



Midpoint of triangle?

Answer 2: Double interpolation



Second interpolation: vector line m0 to P2

$$
\begin{aligned}
P &= sV' + m0 \\
&= s(P2 - m0) + m0 \\
&= sP2 + (1 - s)m0
\end{aligned}
$$

Midpoint at s = 1/3

$$
\begin{aligned}
m &= \frac{1}{3}P2 + \left(1 - \frac{1}{3}\right)m0 \\
&= \frac{1}{3}P2 + \frac{2}{3}m0
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{3}P2 + \frac{2}{3}(0.5P1 + 0.5P0) \\
&= \frac{P2 + P1 + P0}{3}
\end{aligned}
$$

Generalizing: convex combinations of points

A convex combination of a set of points S is a linear combination such that the non-negative coefficients sum to 1

$$C = \sum_{P\ in\ S} \alpha_i P_i \quad \text{with} \quad \sum \alpha_i = 1 \quad \text{and} \quad 0 \leq \alpha_i \leq 1$$

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Is this equation for a line *segment* a convex combination?

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$$P = tP1 + (1 - t)P0$$

Yes. With t in [0,1], t and (1-t) >= 0, and t+(1-t) = 1

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Is this equation for a triangle a convex combination, assuming s and t are in [0,1]?

$$P = sP2 + (1-s)(tP1 + (1-t)P0)$$
$$= sP2 + (1-s)(tP1) + (1-s)(1-t)P0)$$
$$= sP2 + (t-st)P1 + (1-s-t+st)\ P0$$

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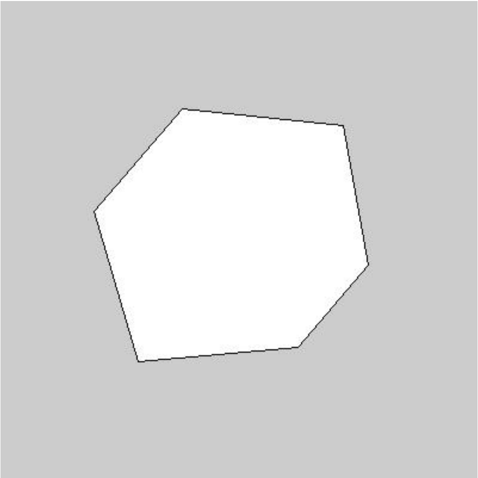
For general polygon, all convex combinations of vertices yields *convex hull*



Linear, affine and convex combinations

Linear: No constraints on coefficients

$$C = \sum_{P\ in\ S} \alpha_i P_i$$

Affine: Coefficients sum to 1

$$C = \sum_{P\ in\ S} \alpha_i P_i \quad \text{with} \quad \sum \alpha_i = 1$$

Convex: Coefficients sum to 1, each in [0,1]

$$C = \sum_{P\ in\ S} \alpha_i P_i \quad \text{with} \quad \sum \alpha_i = 1 \quad \text{and} \quad 0 \le \alpha_i \le 1$$

Linear combinations of points vs. vectors

Point – point yields a ….

Vector – vector yields a …

Point + vector yields a …

Point + point yields a …

Linear combinations of points vs. vectors

Point – point yields a …. vector

Vector – vector yields a … vector

Point + vector yields a … point

Point + point yields a … ???? Not defined

Vectors are closed under addition and subtraction
- Any linear combination valid

Points are not
- Affine combination that sums to 0 yields vector
- Affine combination that sums to 1 yields point
- Convex combination yields point in convex hull

Moral: When programming w/ pts&vtrs, know the output type

What you should know

1. Linear, affine and convex combinations
2. Triangle midpoint