CMSC427 Transformations: Matrix Review

Credit: slides 9+ from Prof. Zwicker

Matrix practice

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad MR = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} =$$

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad RM = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} =$$

$$P = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad MP = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

Matrix practice

$$M = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \qquad MR = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad RM = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad MP = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Matrix transpose and column vectors

$$R^T = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}^T =$$

$$H^T = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}^T =$$

$$P = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

Matrix transpose and column vectors

$$R^T = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

Abstract point of view

- Mathematical objects with set of operations
  - Addition, subtraction, multiplication, multiplicative inverse, etc.
- Similar to integers, real numbers, etc.

But

- Properties of operations are different
  - E.g., multiplication is not commutative
- Represent different intuitive concepts
  - Scalar numbers represent distances
  - Matrices can represent coordinate systems, rigid motions, in 3D and higher dimensions, etc.

Practical point of view

- Rectangular array of numbers

$$\mathbf{M} = \begin{bmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m,1} & m_{2,2} & \dots & m_{m,n} \end{bmatrix} \in \mathbf{R}^{m \times n}$$

- Square matrix if $\mathbf{m} = \mathbf{n}$
- In graphics often $\mathbf{m} = \mathbf{n} = 3, \mathbf{m} = \mathbf{n} = 4$

Matrix addition

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \dots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \dots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & a_{2,2}+b_{2,2} & \dots & a_{m,n}+b_{m,n} \end{bmatrix}$$

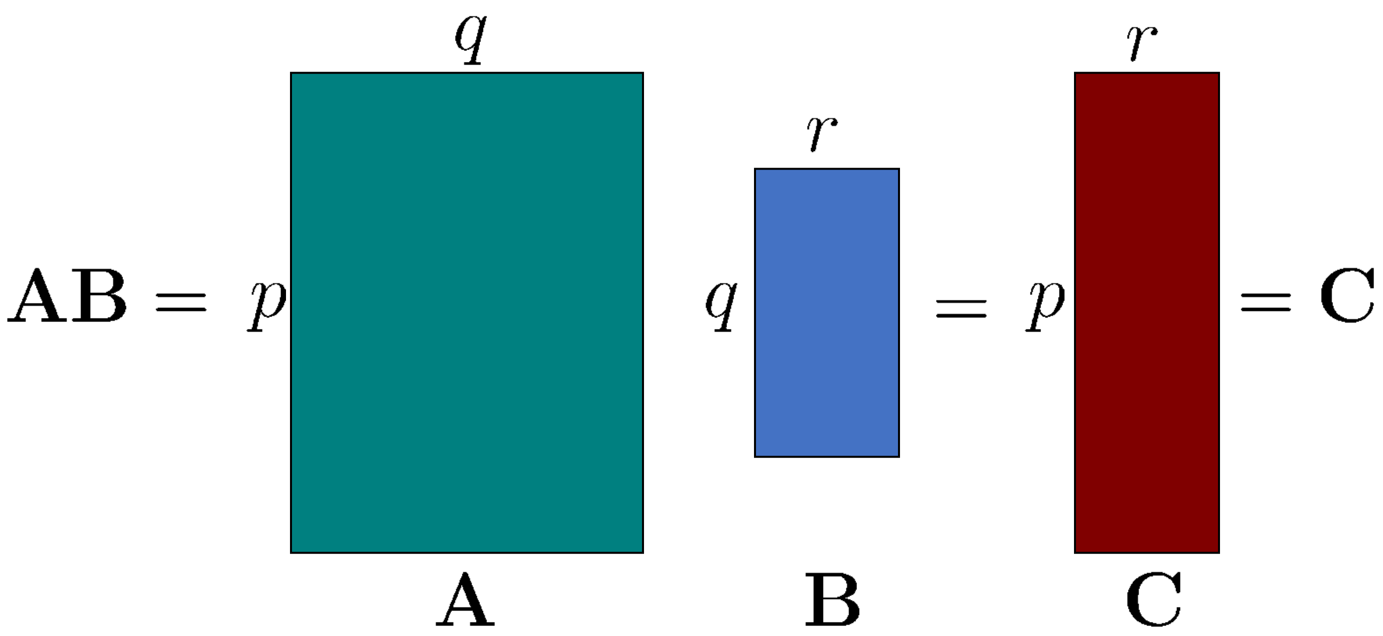
$$\mathbf{A},\mathbf{B}\in\mathbf{R}^{m\times n}$$

Multiplication with scalar

$$s\mathbf{M} = \mathbf{M}s = \begin{bmatrix} sm_{1,1} & sm_{1,2} & \ldots & sm_{1,n} \\ sm_{2,1} & sm_{2,2} & \ldots & sm_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ sm_{m,1} & sm_{2,2} & \ldots & sm_{m,n} \end{bmatrix}$$

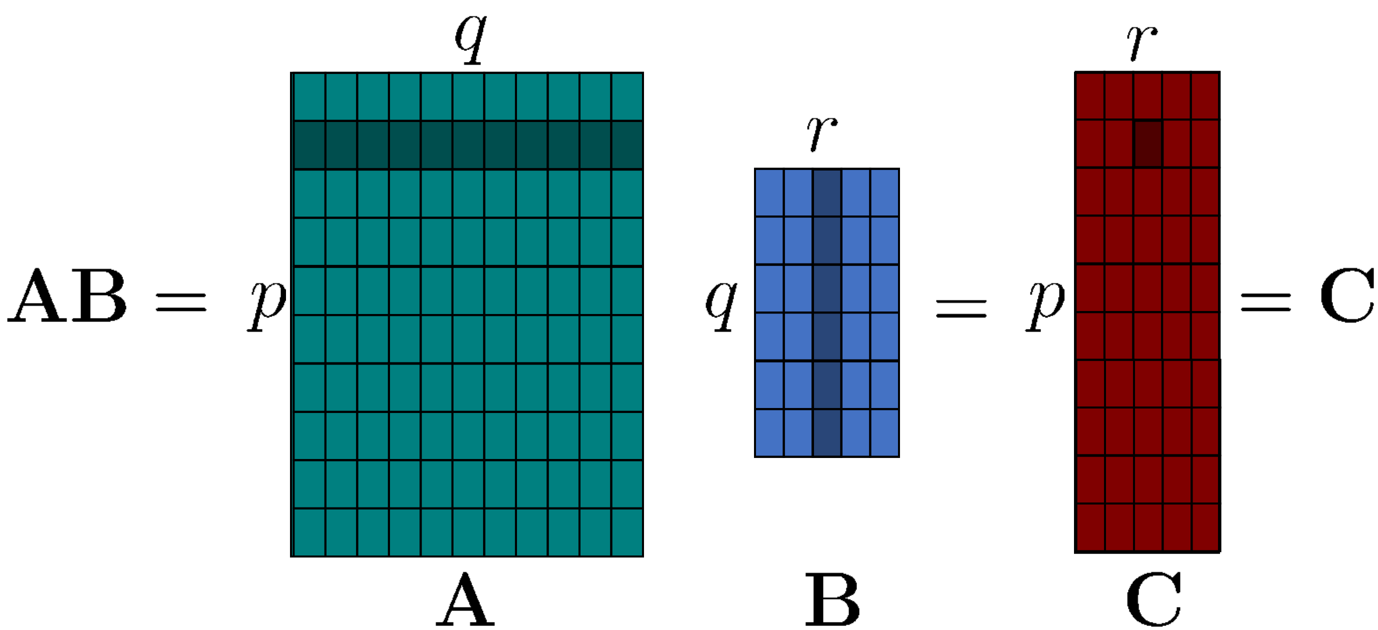
Matrix multiplication

$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$

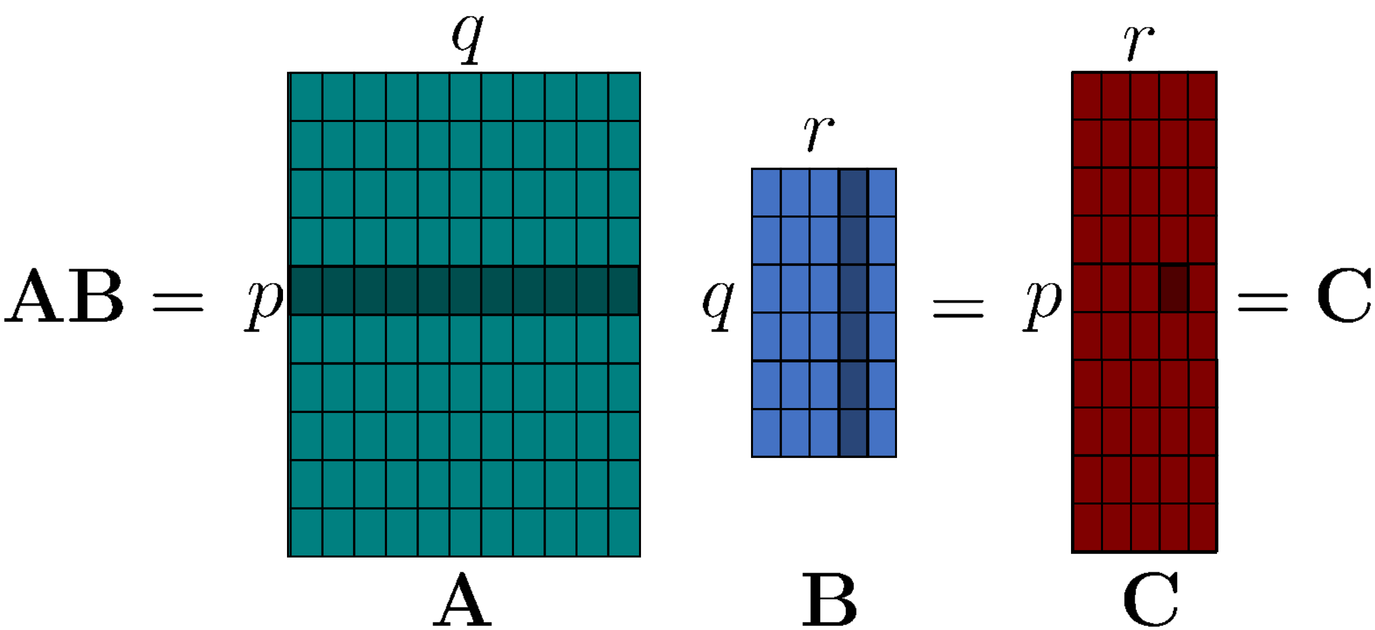


Matrix multiplication

$$\mathbf{A}\mathbf{B} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$



$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$

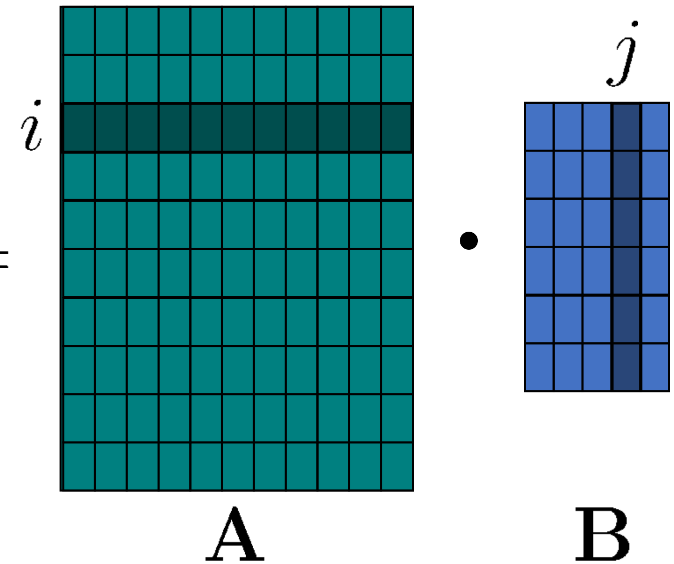


Matrix multiplication

$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$

$$(\mathbf{AB})_{i,j} = \mathbf{C}_{i,j} = \sum_{k=1}^{q} a_{i,k} b_{k,j}, \quad i \in 1..p, j \in 1..r$$

$$(\mathbf{AB})_{i,j} = \mathbf{C}_{i,j} =$$

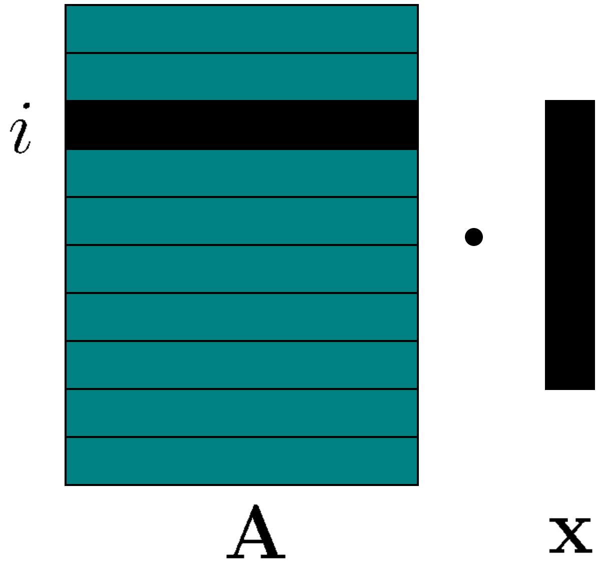


Special case: matrix-vector multiplication

$$\mathbf{A}\mathbf{x} = \mathbf{y}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{x} \in \mathbf{R}^{q}, \mathbf{y} \in \mathbf{R}^{p}$$

$$(\mathbf{A}\mathbf{x})_i = \mathbf{y}_i = \sum_{k=1}^{q} a_{i,k} x_k$$

$$(\mathbf{A}\mathbf{x})_i = \mathbf{y}_i =$$



Linearity

- Distributive law holds

  i.e., matrix $\mathbf{A}(s\mathbf{B} + t\mathbf{C}) = s\mathbf{AB} + t\mathbf{AC}$.

  http://en.wikipedia.org/wiki/Linear_map

- But multiplication is not commutative,

  in general

$$\mathbf{AB} \neq \mathbf{BA}$$

Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbf{R}^{n \times n}$$

$$\mathbf{MI} = \mathbf{IM} = \mathbf{M}, \quad \text{for any } \mathbf{M} \in \mathbf{R}^{n \times n}$$

Definition

If a square matrix M is non-singular, there exists a unique inverse M^{-1} such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

- Note

$$(\mathbf{MPQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\mathbf{M}^{-1}$$

- Computation
  - Gaussian elimination, Cramer's rule (OctaveOnline)
  - Review in your linear algebra book, or quick summary http://www.maths.surrey.ac.uk/explore/emmaspages/option1.html

Java vs. OpenGL matrices

- OpenGL (underlying 3D graphics API used in the Java code, more later)
  http://en.wikipedia.org/wiki/OpenGL
  - Matrix elements stored in array of floats float M[16];
  - "Column major" ordering

$$\begin{bmatrix} m[0] & m[4] & m[8] & m[12] \\ m[1] & m[5] & m[9] & m[13] \\ m[2] & m[6] & m[10] & m[14] \\ m[3] & m[7] & m[11] & m[15] \end{bmatrix}$$

- Java base code
  - "Row major" indexing
  - Conversion from Java to OpenGL convention hidden somewhere in basecode!

$$\begin{bmatrix} m(0,0) & m(0,1) & m(0,2) & m(0,3) \\ m(1,0) & m(1,1) & m(1,2) & m(1,3) \\ m(2,0) & m(2,1) & m(2,2) & m(2,3) \\ m(3,0) & m(3,1) & m(3,2) & m(3,3) \end{bmatrix}$$