CMCS427 Dot product review

Computing the dot product

The dot product can be computed via

a) Cosine rule	$a \bullet b = \lvert a\rvert\lvert b\rvert \cos\theta$
b) Coordinate-wise	$a \bullet b = ax * bx + ay * by$

Exercises:

1) If $a \bullet b$, $\lvert a\rvert$ and $\lvert b\rvert$ all equal 1, what's the angle between the vectors?

Since $a \bullet b = \lvert a\rvert\lvert b\rvert \cos\theta$ becomes $1 = \cos\theta$, then $\theta = 0$

2) If $a=<2,-1>$ and $b=<3,2>$, what is $a \bullet b$ and what is the angle between the vectors?

$a \bullet b = <2,-1> \bullet <3,2> = 4$ $\quad$ $\lvert a\rvert = 2.2361$, $\quad$ $\lvert b\rvert = 3.6056$

$\theta = \text{acos}(\ 4/(2.2361*3.6056)\ ) = 1.0517$ radians $= 60.26$ degrees

3) If $v \bullet w = \frac{1}{2} \lvert v\rvert\lvert w\rvert$ what is the angle between u and v?

This gives $(v \bullet w)/(\ \lvert v\rvert\lvert w\rvert) = \frac{1}{2}$, so $\text{acos}(1/2) = 1.0472$ radians, 60 degrees

Properties of the dot product (dot product algebra)

The dot product is

a) Commutative	$a \bullet b = b \bullet a$
b) Distributive	$a \bullet (b+c) = a \bullet b + a \bullet c$
c) Non-canceling	$a \bullet b = a \bullet c$ does not mean $b=c$ (it does give $a \bullet (b-c)=0$)
d) Magnitude squared	$a \bullet a = \lvert a\rvert^2$
e) Linear with scalars	$sa \bullet b = s(a \bullet b)$
f) From (b) and (d), linear	$a \bullet (sb+rc) = s(a \bullet b) + r(a \bullet c)$
g) Zero if orthongal	$a \bullet b = 0$ if a is orthogonal to b

Example: Rewrite the magnitude of $\lvert a-b\rvert^2$ to eliminate a-b

$$\lvert a-b\rvert^2 = (a-b)\bullet(a-b) = a \bullet a - a \bullet b - b \bullet a + b \bullet b = a \bullet a - 2(a \bullet b) + b \bullet b = \lvert a\rvert^2 - 2(a \bullet b) + \lvert b\rvert^2$$

Exercises:

1) Following the example, rewrite $\quad \lvert 2a-b\rvert^2$

$\lvert 2a-b\rvert^2 = (2a-b)\bullet(2a-b) = 2a \bullet 2a - 2a \bullet b - b \bullet 2a + b \bullet b = 2(a \bullet a) - 4(a \bullet b) + b \bullet b = 2\lvert a\rvert^2 - 4(a \bullet b) + \lvert b\rvert^2$

2) If v is orthogonal to u and w separately, is it orthogonal to u+w?

Yes. $v \bullet (u+w) = v \bullet u + v \bullet w = 0 + 0 = 0$

3) Simplify the following:

a) $(v-w)\bullet w + v\bullet w$ **$= v\bullet w - w\bullet w + v\bullet w = 2(v\bullet w) - |w|^2$**

b) $(v+w)\bullet v - (v+w)\bullet w$ **$= v\bullet v + v\bullet w - v\bullet w - w\bullet w = v\bullet v - w\bullet w = |v|^2 - |w|^2$**

c) $(v+w)\bullet(v+w) - 2v\bullet w$

$= v\bullet v + v\bullet w + w\bullet v + w\bullet w - 2v\bullet w$
$= v\bullet v + v\bullet w + v\bullet w + w\bullet w - 2v\bullet w$
$= v\bullet v + 2v\bullet w - 2v\bullet w + w\bullet w$
$= |v|^2 + |w|^2$

Using sign of the dot product for testing angles
If $a\bullet b = 0$, the angle between the vectors is zero
If $a\bullet b > 0$, the angle is acute (less than 90 degrees)
If $a\bullet b < 0$, the angle is obtuse (greater than 90 degrees)
Note: to use this test you generally need a cut off for equal to zero, so $|a\bullet b| < \text{delta}$, delta small

Exercises:
1) Test the following pairs: $\langle 2,1\rangle$ and $\langle -1,1\rangle$; $\langle 2,0\rangle$ and $\langle 0,1\rangle$; $\langle 3.4,-3.7\rangle$ and $\langle 3.7,3.3\rangle$.

Pair 1: $\langle 2,1\rangle \bullet \langle -1,1\rangle = -1$, angle is greater than 90
Pair 2: $\langle 2,0\rangle \bullet \langle 0,1\rangle = 0$, angle is 0
Pair 3: $\langle 3.4,-3.7\rangle \bullet \langle 3.7,3.3\rangle = 0.37$, angle is less than 90

2) Would the length of the vectors a and b influence the numeric sensitivity of the test? Eg, if a and b are both long vectors, does it make a difference relative to your delta cut off?

Yes, the length does influence the dot product. $(sa)\bullet b = s(a\bullet b)$, so if $s \gg 1$, the dot product is increased. (This is why we often use normalized vectors for such tests).

2) The perp vector in 2D is given by $v\_perp = \langle -y,x\rangle$ for the vector $v = \langle x,y\rangle$.

With 3D vectors it isn't as easy to define a single perp vector - there's an infinity of directions rotating around a vector that are candidates. But, given a 3D vector $w=\langle x,y,z\rangle$, can you define a perpendicular vector by inspection that's like the perp vector? A vector w_perp such that $w\_perp\bullet w = 0$?

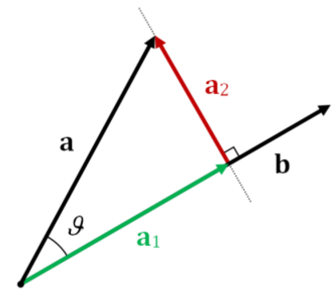
There's multiple options. One is $\langle -y,x,0\rangle$ so $\langle x,y,z\rangle\bullet\langle -y,x,z\rangle = 0$

Dot product and projection

A number of critical applications of the dot product are from using it to find projections.

• The *vector projection* of a onto b is the vector a1 which is the component of a in the direction of b.

• The s*calar projection* of a onto b is the length of a1.



A starting equation is $a_1 = \frac{a \bullet b}{b \bullet b} b$ and $|a_1| = \left|\frac{a \bullet b}{b \bullet b} b\right|$

From this equation we have, if |b|=1, that $a_1 = (a \bullet b)\hat{b}$

This gives $|a_1| = |(a \bullet b)||\hat{b}| = |(a \bullet b)|$

Examples:

• What is the projection of <2,3,4> onto <0,1,0>? <0,3,0> with a length of 3. Projections onto the axes are the vector's coordinate of that axis.

• What is the projection of <1,2> onto <1,1>?

$$a_1 = \frac{a \bullet b}{b \bullet b} b = \frac{< 1,2 > \bullet < 1,1 >}{< 1,1 > \bullet < 1,1 >} < 1,1 > = \frac{3}{2} < 1,1 > = < 1.5,1.5 >$$

The length is |<1.5,1.5>| = sqrt(2*(1.5*1.5)) = 2.1213

• What is the projection of <0.5,0.5> onto <2,3>?

$$a_1 = \frac{a \bullet b}{b \bullet b} b = \frac{< 0.5,0.5 > \bullet < 2,3 >}{< 2,3 > \bullet < 2,3 >} < 2,3 > = \frac{2.5}{13} < 2,3 > = < 0.38462, 0.57692 >$$

The vector projection has the following applications (among others):

• Vector rejection. The vector a2 (see above) that is the component of a orthogonal to b.

• Coordinate frame resolution. The vectors a1 and a2 are orthogonal and represent an alternative coordinate frame to the standard x,y frame (in 2D)

• Distance from a point to a line. The distance from a point p to a line defined by point-vector.

Details of each application.

• Vector rejection

Once you have a1, then a = a1+a2 so a2 = a – a1.

Example: What is the vector rejection of <1,2> onto <1,1>? Since a1 = <1.5,1.5>, then

a2 = <1,2> - <1.5,1.5> = <-0.5,+0.5> and a1•a2 = 0

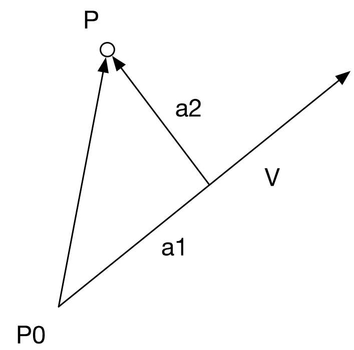
• Coordinate frame resolution

Once you have a1 and a2, you have a pair of orthogonal vectors. If you normalize then you have an orthonormal basis for a new coordinate system (vectors are orthonormal if orthogonal and unit length). Now

$$\widehat{a_1} = \frac{a1}{|a1|} = \frac{<1.5,1.5>}{\sqrt{1.5^2+1.5^2}} =< .70711, 0.70711 >, \text{ and } \widehat{a_2} =< -0.70711, +0.70711 >$$

• Distance from a point to a line

If we're given a line in point-vector form with P0 and v, then the distance from an arbitrary point P to the line is given by the length of the vector rejection a2, with a = P-P0



$$d = |a_2| = \left|(P - P0) - \frac{(P - P0) \bullet v}{v \bullet v} v\right|$$

If v is of unit length, then

$$|a_2| = |(P - P0) - ((P - P0) \bullet v) \bullet v|$$

Example: What is distance of the point (3,4) from the line defined by P0=(0,0) and v=<1,1>?

$$d = \left|(P - P0) - \frac{(P - P0) \bullet v}{v \bullet v} v\right|$$

$$d = \left|(< 3,4 > - < 0,0 >) - \frac{(< 3,4 > - < 0,0 >) \bullet < 1,1 >}{< 1,1 > \bullet < 1,1 >} < 1,1 >\right|$$

$$d = \left|< 3,4 > - \frac{7}{2} < 1,1 >\right| = |< 3,4 > - < 3.5,3.5 >| = |< -0.5,0.5 >| = 0.70711$$

Exercises:

1) What's the vector a1 and scalar projection of a=<3,2,1> onto b=<0,2,2>?

$$\boldsymbol{a_1} = \frac{\boldsymbol{a \bullet b}}{\boldsymbol{b \bullet b}}\boldsymbol{b} = \frac{<3,2,1>\bullet<0,2,2>}{<0,2,2>\bullet<0,2,2>} < \mathbf{0, 2, 2} > = \mathbf{0.75} < \mathbf{0, 2, 2} > = < \mathbf{0, 1.5, 1.5} >$$

|a1| = |$< \mathbf{0, 1.5, 1.5} >$ | = 2.1213

2) What the vector rejection a2 for this example?

a2 = a – a1 = <3,2,1> - <0,1.5,1.5> = <3,0.5,-0.5> **Note that a1•a2 = 0**

3) And, putting (1) and (2) together, what's the normalized coordinate frame based on a1 and a2 for this case?

Normalize the two vectors a1 and a2 (call them A1 and A2 for ease of typing.)

A1 =$< \mathbf{0, 1.5, 1.5} >$/ 2.1213	**= < 0,**	**0.70711,**	**0.70711>**
A2 = <3,0.5,-0.5>/3.0822	**= < 0.97333,**	**0.16222,**	**-0.16222>**

4) What's the distance from the point (7,3) to the line given by P0=(1,2) and v=<2,1>?

$$\boldsymbol{d} = \left|(\boldsymbol{P} - \boldsymbol{P0}) - \frac{(\boldsymbol{P} - \boldsymbol{P0}) \bullet \boldsymbol{v}}{\boldsymbol{v} \bullet \boldsymbol{v}} \boldsymbol{v}\right|$$

$$\boldsymbol{d} = \left|((\mathbf{7}, \mathbf{3}) - (\mathbf{1}, \mathbf{2})) - \frac{((\mathbf{7}, \mathbf{3}) - (\mathbf{1}, \mathbf{2})) \bullet < \mathbf{2}, \mathbf{1} >}{< \mathbf{2}, \mathbf{1} > \bullet < \mathbf{2}, \mathbf{1} >} < \mathbf{2}, \mathbf{1} >\right|$$

= 1.7889

5) There's another equation for the distance from a point to a line based on the normal vector (hint: consider the perp vector if you have a line in point-vector form.) Develop an equation for the distance from a point P to a line defined by P0 and v, and then apply it to the data in (4).

If the line is defined by P(t) = P0 + tv, then the normal vector is v_perp.

Let n be a normalized version of v_perp: n = v_perp/|v_perp|

The point-normal form for a line is n•(P-P0) = 0.
Here the projection of P-P0 is zero, so the distance from the point to the line is zero.

This same equation gives the distance of P from the line, so is what we want (but for distance, we need the absolute value | n•(P-P0) |

For this case, with P=(7,3) and P0=(1,2) and v=<2,1>, we have

n = <-1,2>/|<-1,2>| = <-0.44721, 0.89443>

d = |n•(P-P0)| = |<-0.44721, 0.89443> • ((7,3)-(1,2)) |= 1.7889

Some of these exercises came from the file below. It has exercises with solutions, so is good for self-study.   https://dnichols30582.edublogs.org/files/2015/12/11.6-Solutions-1eupr84.pdf