CMSC427 Fall 2020
Exercises for Tuesday, Nov. 17th
Rotations and quaternions

This lecture will be on quaternions and their applications in representing rotations. The gist of the presentation will be to explain how quaternions are a mathematically clean way of encoding angle-axis representations of rotation.

Objectives. You should be able to:
1. List methods for representing rotations: Euler angles, orthonormal rotation matrices, two vectors, angle axis and quaternions.
2. Explain how unit vectors in the 2D plane represent direction, or alternatively object pose, and have only one degree of freedom (1 DOF) (an angle θ).
3. Explain how unit vectors in 3D still represent direction, but have 2 DOF and aren't enough to represent a 3D rotation.
4. Explain how adding an angle to a 3D vector for angle axis allows you to represent 3D rotations since you now have 3 DOF.
5. Work out the averaging of angle-axis rotations to show you can average them but need to renormalize to maintain a unit vector – hence interest in a cleaner formulation, quaternions.
6. Multiply the quaternion units i, j, k by the Hamiltonian rules, with attention to cross products
7. Represent and manipulate quaternions using the rules for the unit vector.
8. Multiply two quaternions by term-wise units and by the multiplication formula.
8. Conjugate a quaternion and compute qq*, and show that (pq)* = q*p*.
9. Compute the length of a quaternion.
10. Work with unit length quaternions, including pure quaternions.
11. Compute rotation by angle-axis using Rodrigues' formula.
12. Show that a vector rotated by quaternion q using qvq* is equivalent to Rodrigues' formula.
13. Explain gimbal lock and how quaternions are better than Euler angles in handling it.
14. Explain the slerp formula for smoothly interpolating between quaternions.
15. Show that if p and q are "rotation" quaternions, then pq is the combined rotation.

Elms modules:
See Week 10, lecture for Tuesday Nov. 10th .

Readings:
This is a lightweight article that gives the historical context of quaternions and how important they are to mathematics, including the ordinary vectors and vector operations we're familiar with.
https://www.quantamagazine.org/the-strange-numbers-that-birthed-modern-algebra-20180906/

Our main source will be lecture 23 from David Mount, posted to Elms.

This presentation from John Huerta of Fullerton College explains how quaternions relate to complex numbers.
https://math.ucr.edu/~huerta/introquaternions.pdf

We are not doing the two vector representation in detail, but this page give the equation if you are interested:
https://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula

There are some online animations of quaternions:
https://quaternions.online
https://eater.net/quaternions

If you want to dive deeper into quaternions there are a number of sources. Shoemake gives a relevant explanation of animations with quaternions.
https://www.cs.cmu.edu/~kiranb/animation/p245-shoemake.pdf
https://web.mit.edu/2.998/www/QuaternionReport1.pdf

Exercises:

1. Compute the average of the two angle-axis rotations (angle=0, vector=<1,0,0>) and (angle=π/4, vector=<0,1,1>) and renormalize.
2. Compute the product of the two quaternions p=< 2, 1, 1, 0> and q=<1,0,1,1>.
3. Compute the conjugate and length of p. What is the inverse of p?
4. Work out for yourself that pp* = a2 + b2 + c2 + d2
5. Give two unit vectors in the same family as the pure unit quaternion p=< 0,1,1,0>
6. Work out the rotation of a vector by angle-axis (angle= π/4, vector=<1,0,0>) using Rodrigues' formula.
7. Work out for yourself the multiplication of two quaternions p, q and show it fits the standard formula with dot and cross products.
8. What is the quaternion for a rotation by angle theta around the x-axis? The z-axis?
9. Compute the combine rotation from a 45 degree rotation around the x-axis followed by a 90 degree rotation around the z-axis (use #8 to find the quaternions involved.)