CMSC427 Final
Fall 2020

University honor pledge. I pledge on my honor that I have not given nor received any unauthorized assistance on this assignment/examination.

Signature: SOLUTIONS

Instructions:

• Partial credit will be given, which is easiest when you write clearly, explain your answers, and show the equations you use.

• You shouldn't need a calculator for calculations – setting up equations will be enough – but may use one if you wish. If you use Octave, Wolfram Alpha, or another site to compute note that – but the exam should not require it.

• The exam is open book and open notes. You may use the internet, but the boundary is that you may not copy answers from websites or others.

Grade

Section	Worth	Received
I. Short answer	20	
II. Surfaces and shading	10	
III. Parametric curves	15	
IV. Fractals	15	
V. Meshes and winged edge	15	
VI. Ray tracing	20+5	
VII. Procedural textures	5	
Total	105	

Section I: Short answer (20 pts)

1. (5 pts) Show that this implicit function is satisfied by the parametric equations

$$(x-a)^2 + (y-b)^2 = r^2$$

$$Px(t) = r\ \cos(2\pi t) + a$$
$$Py(t) = r\ \sin(2\pi t) + b$$

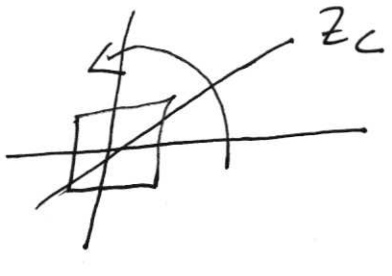
Substitute into implicit equation

$$(x-a)^2 + (y-b)^2 = (P_x(t) - a)^2 + (P_y(t) - b)^2$$
$$= (r\cos 2\pi t + a - a)^2 + (r\sin 2\pi t + b - b)^2$$
$$= r^2 \cos^2 2\pi t + r^2 \sin^2 2\pi t$$
$$= r^2$$
$$\therefore$$

2. (5 pts) In the camera coordinate system calculations you start with at, lookat and up. What happens if up is parallel to Zc = at – lookAt?

If up $\parallel$ Zc then up $\times$ Zc $= \vec{0}$

and x_c, y_c are not defined.



The rotation of the camera around the camera axis ~~th~~ is not defined. The purpose of the up vector is to give this rotation.

3. (3 pts) What is the quaternion for a rotation of 180 degrees around the x-axis? Give in vector format q = <w,x,y,z> and in terms of basis elements i, j and k.

$$q = \langle \cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{U}\rangle \text{ with } \vec{U} = \langle 1,0,0\rangle$$

$$= \langle \cos 90^\circ, \sin 90^\circ \langle 1,0,0\rangle\rangle$$

$$= \langle 0, 1, 0, 0\rangle$$

$$= i$$

4. (7 pts) Use the quaternion from (3) to rotate the vector v = <0,1,0>. Show how you set it up, show the calculations, and give a final answer.

Rotation of v by q is qvq^*

$q = i$ $\quad v = j$

$q^* = -i$

OR

$q = 0 + i + 0j + 0k$

$v = 0 + 0i + j + 0k$

$q^* = 0 - i - 0j - 0k$

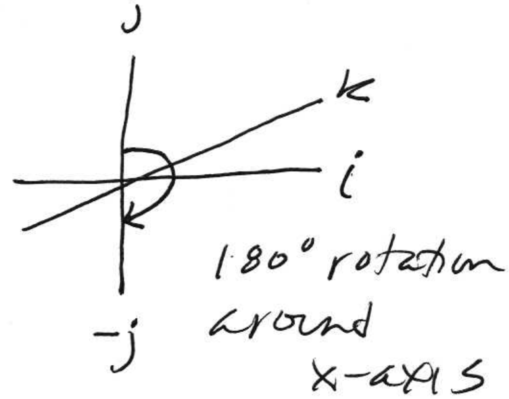
$$qvq^* = i * j * -i$$

$$= (i * j) * -i$$

$$= k * -i$$

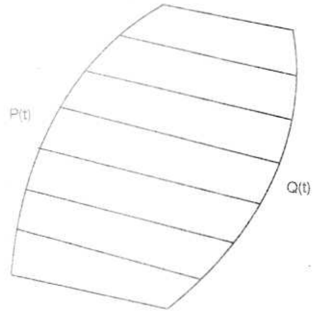
$$= -(kj) = -j$$

So result is $\langle 0, 0, -1, 0\rangle$



Section II: Surfaces and shading (10 pts)

1.(3 pts) Interpolating patch. Assume you have two curves, P(t) and Q(t) that draw curves in three dimensions. How would you recreate a patch by linear interpolation between the two?



lerp(s, P(t), Q(t))

or

$$P(s,t) = (1-s)P(t) + sQ(t)$$

4. (7 pts) Compute the three vectors of the Frenet frame for the parametric curve p(t) = < 3, 2t^2, cos t > with the tangent, normal and binormal. Don't normalize for this exercise.

$P'(t) = \langle 0, 4t, -\sin t \rangle$ Tangent

$P''(t) = \langle 0, 4, -\cos t \rangle$ Normal

$$P'(t) \times P''(t) = \begin{vmatrix} i & j & k \\ 0 & 4t & -\sin t \\ 0 & 4 & -\cos t \end{vmatrix} = \langle -4t\cos t + 4\sin t, 0, 0 \rangle$$

Binormal

Section III: Parametric curves (15 pts)

1. (5 pts) If we add a tuning parameter α to our equations, that tunes the influence of the tangent components T0 and T1, how does it change the curve? Set α=0 and work out the new P(t) as a polynomial – what curve do you get? If we make α larger, what happens? Give an informal statements of the changes.

$$P(t) = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} 2 & -2 & \alpha & \alpha \\ -3 & 3 & -2\alpha & -\alpha \\ 0 & 0 & \alpha & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{P0} \\ \boldsymbol{P1} \\ \boldsymbol{T0} \\ \boldsymbol{T1} \end{bmatrix}$$

$$P(t)\Big|_{\alpha=0} = [t^3 t^2 t\ 1]\begin{bmatrix} 2 & -2 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P0 \\ P1 \\ T0 \\ T1 \end{bmatrix}$$

$$= [t^3 t^2 t\ 1]\begin{bmatrix} 2P0 - 2P1 \\ -3P0 + 3P1 \\ 0 \\ P0 \end{bmatrix}$$

$$= t^3(2P0 - 2P1) + t^2(-3P0 + 3P1) + P0$$

This is a line as multiples of $-P0 + P1$

$$= (P0 - P1)\,2t^3 + (P0 - P1)(-3t^2) + P0$$

$$= (3t^2 - 2t^3)(P1 - P0) + P0$$

And the tangent has no contribution.

As α increases, the influence of the tangents increase and the curve sharpens to meet the tangent direction faster

2. (10 pts) Using the derivation of the Hermite cubic curve as done in the lecture materials, apply that derivation to create a similar parametric form of a Hermite quadratic curve. Given three points P0, P1, P2, calculate a quadratic form of the Hermite curve. Have the curve interpolate through P0 and P1, and use P2-P1 as the tangent for the second endpoint. Give as your final answer the new (3x3) characteristic matrix for the curve. Unlike the instructor, read the points carefully for your derivation.

There are multiple approaches to the solution. The constraints are given by

$$P(t) = at^2 + bt + c$$

$$P'(t) = 2at + b$$

At $t=0$: $P(0) = a0^2 + b0 + c = c = x_0$

At $t=1$: $P(1) = a + b + c = x_1$

$P'(1) = 2a + b = \Delta x = x_2 - x_1$

Version 1: $\begin{bmatrix} x_0 \\ x_1 \\ \Delta x \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ x_1 \\ x_2 - x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Inverting M (valid solution) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}}} \begin{bmatrix} x_0 \\ x_1 \\ x_2 - x_1 \end{bmatrix}$

Version 2: (Also valid) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$

$$= \underline{\underline{\begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix}}} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

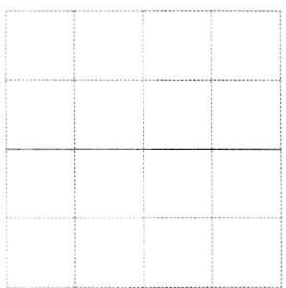
Can also solve for the second solution directly.

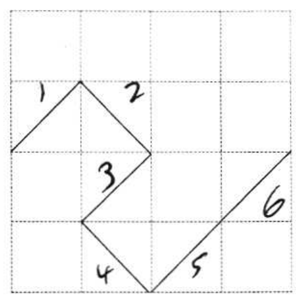
Section IV: Fractals (15 pts)

1. (5 pts) In an Iterated Function System (IFS), self-similarity comes from what part of the system?

The mappings of an IFS duplicate and usually shrink the overall shape, so create self-similarity.

2.(10 pts) Given this diagram of recursive step of a possible L-system curve, with the initiator on the left and the next step on the right, answer the questions below.





a) Give an L-system with grammar for drawing the shape defined by the curve?

$\delta = 45^\circ$

F initiator

$F \to +F--F--F++F++FF-$

starts and ends horizontal

$\delta = 90^\circ$

$F = F-F-F+F+FF$

b) Give the fractal dimension of the curve.

$N = 6$ six parts

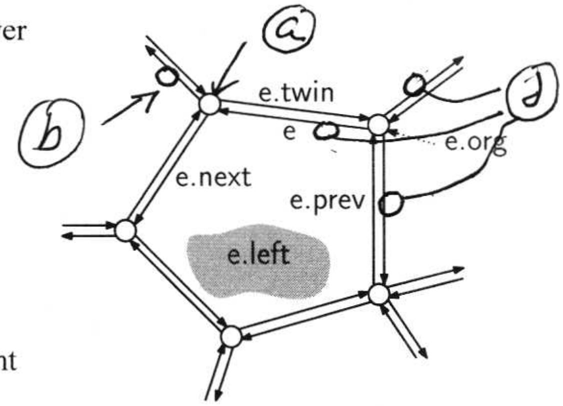
$s = \frac{1}{4}\sqrt{2} = \frac{1}{2\sqrt{2}}$

$$D = \frac{\log N}{\log 1/s}$$

$$= \frac{\log 6}{\log 1/\frac{1}{2\sqrt{2}}} = \frac{\log 6}{\log 2\sqrt{2}}$$

$$\doteq \underline{1.72}$$

Section V: Meshes and winged edge (15 pts)



The DECL representation is given on the right. Given this, answer the questions below.
Give short expressions to give:
a) e.incident: the vertex an edge points **to**

e. twin. org

b) e.nextOuter: the edge that comes after e if you are traversing face e.right in a clockwise direction (ie, the edge out of e.incident that isn't e.next or e.twin)

e. twin. prev. twin
or e. next. twin. next

Give algorithms to
c) List the vertices of the *left* face of e in clockwise order.

```
curr = e
output e.org
curr = e.prev
while (curr != e) {
    output curr.org
    curr = e.prev.
}
```

OR

```
curr=e
do {
   output curr.org
   curr = e.prev
} until(curr==e)
```

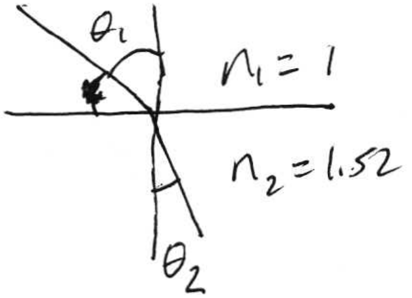
d) List the edges originating from e.org in any order.

e.org is a vertex, and each vertex connects to one edge. Assume that is e. We don't know how many edges orginate from e.org so ~~need~~ loop

```
curr = e
do {
  output curr
  curr = e.twin.next
} until (curr==e)
```

Section VI: Ray Tracing (20 pts)

1. (5 pts) Refraction. Assuming you have a light ray leaving air and hitting crown glass with an angle of incidence of 30 degrees, with a refraction index of 1.52 for the glass and 1.00 for the air, what is the transmission angle of the refracted ray?



$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2} \Rightarrow \sin\theta_2 = \frac{n_1}{n_2}\sin\theta_2$$

$$\theta_2 = \arcsin\left(\frac{1}{1.52}\sin 30^\circ\right)$$

$$\cong 0.335 \text{ rad} \cong 19.1 \text{ degrees}$$

2. (5 pts) Reflection. With the data from problem 1, what is the percentage of light that would be reflected (you don't need to compute the reflection angle or direction)?

Schlichs approximation:

$$R(\theta) = R_0 + (1-R_0)(1-\cos\theta)^5$$

$$R_0 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$R_0 = \left(\frac{1.52-1.0}{1.52+1}\right)^2 = \left(\frac{0.52}{2.52}\right)^2 \cong 0.04257$$

$$R(30^\circ) \cong 0.042 + (1-0.042)(1-0.866)^5$$

$$\cong 0.042 + \underline{0.000042}$$

$$\cong 0.042$$

↳ small term.

3. (10 pts) Intersections. One approach to computing ray-cylinder intersection is to translate, scale and rotate the ray and cylinder so the cylinder is in a normalized position, compute the intersection, and then transform the hit point back. Assume you have a normalized cylinder defined as follows:

Central axis along y-axis, going up, with P0 at the origin and P1 at (0,1,0). Radius 1.

And a ray defined by p(t) = q + vt.

Set up the computation of the hit point for this case, and once you have the hit point, the normal.

The question didn't make clear if the endcap discs were required.

For infinite cylinder

$x^2 + z^2 = 1$ ray $P(t) = P_0 + tv$

$(P_x(t))^2 + (P_z(t))^2 = 1$

$(P_0^x + tv_x)^2 + (P_0^z + tv_z)^2 = 1$

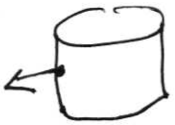
$P_0^{x^2} + 2P_0^x v_x t + v_x^2 t^2 + P_0^{z^2} + 2P_0^z v_z t + v_z^2 t^2 = 1$

$(v_x^2 + v_z^2)t^2 + (2P_0^x v_x + 2P_0^z v_z)t + (P_0^{x^2} + P_0^{z^2} - 1) = 0$

Solve quadratic with $a = (v_x^2 + v_z^2)$, $b = (2P_0^x v_x + 2P_0^z v_z)$, and $c = (P_0^{x^2} + P_0^{z^2} - 1)$

If solution exists, take smaller of t's, substitute into $P(t) = P_0 + tv$ for hit point

Normal is $\langle x, 0, z \rangle$ for hit point $\langle x, y, z \rangle$



If you add in endcaps, then you need to intersect the ray with planes $y=0$ and $y=1$

For $y=0$, $P_y(0) = P_0^y + v_y t = 0$

t_0 so $t = \dfrac{-P_0^y}{v_y}$ if $v_y \neq 0$.

For $y=1$, $P_y(0) = P_0^y + v_y t = 1$

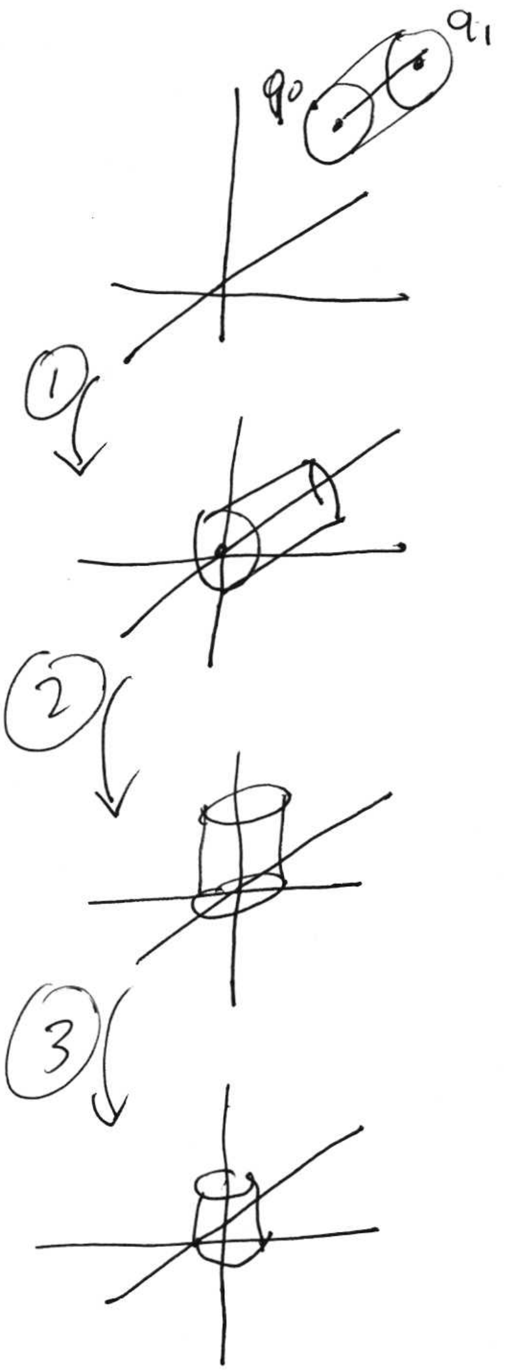
t_1 so $t = \dfrac{1-P_0^y}{v_y}$ if $v_y \neq 0$

Given t_0 and t_1, substitute into $P(t)$, find the hit point, and see if ~~[illegible]~~ distance from y-axis is ≤ 1 (which is $P_x^2(t) + P_y^2(t) \leq 1$)

Pick the smallest of t_0, t_1 and the cylinder hit t_c.

Normals are $\langle 0,1,0 \rangle$ for upper disc, $\langle 0,-1,0 \rangle$ for lower.

4. (5 pts) Challenge question. We have done everything needed to solve this problem, but have not done specifically this sequence of steps. Assume you have an arbitrarily oriented cylinder in space defined by two points q0 and q1, and a radius r. How would you translate, rotate and scale it to a standard position with q0 at (0,0,0), q1 at (0,1,0), and the new radius at 1? You don't have to worry about degenerate cases (q0=q1, r=0, and so on), and you don't have to give matrices, just clear versions of each transformation.



Step one: translate
p_0 to the origin

Step two: rotate
cylinder to align with
y-axis. Use angle axis
with axis $(p_1 - p_0) \times \langle 0,1,0 \rangle$
and angle by sin of cross
product (sin equation for cross)

Step three: scale x, z
by radius and y by $\frac{1}{r}$

Section VII: Procedural Textures (5 pts)

1. (5 pts) Given the definition of Perlin noise from lecture, give an approach on how you would pick p to make it follow a Pink noise frequency plot. Then, do the same for White noise. Extra credit – adapt it to make Brown noise.

$$\text{perlin}(t) = \sum_{i=0}^{k} p^i \text{noise}(2^i t)$$

The frequency for each term in the summation is $f = 2^i$, the amplitude $a = p^i$

white noise the amplitude is constant so $a = 1^i$ with $p = 1$ works

pink noise the amplitude scales with $1/f$ so $p = \frac{1}{2}$ => $\left(\frac{1}{2}\right)^i \text{noise}(2^i t)$

brown noise the amplitude scales with $1/f^2$ so $p = \left(\frac{1}{2}\right)^2$ gives $\left(\frac{1}{2}\right)^{2i} \text{noise}(2^i t)$