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A new characterization of the majority rule

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Abstract

A seminal paper by May [Econometrica 20 (1952) 680] characterizes the majority rule in terms of anonymity, neutrality, and positive responsiveness. Whereas the former two axioms are natural and fairly weak, the positive responsiveness axiom is usually criticized for being too strong. In this note, we provide an alternative characterization of the majority rule in terms of neutrality, Pareto optimality, and a new axiom that we call reducibility to subsocieties.

Keywords: IIA; Majority rule; Pareto

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1. Introduction

Given a society of n individuals that is to choose between a pair of two alternatives, what is an appropriate method or function that assigns a social preference to every profile of individual preferences? In a mathematical formulation, let x and y be two alternatives, and let $S = \{1, 2, \ldots, n\}$ denote a society with n individuals called *voters*. Every voter $i \in S$ has a *preference* $R_i \in \{-1, 0, 1\}$ over the two alternatives x and y. Here $R_i = 1$ means that voter i prefers x to y, $R_i = 0$ means that voter i is indifferent, and $R_i = -1$ means that voter i prefers y to x. These preferences are collected in the *profile* vector $R = (R_1, \ldots, R_n) \in \{-1, 0, 1\}^n$. A *social welfare function* is a function

$$F: \bigcup_{n>0} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$$

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which gives an *aggregate* preference for any preference profile of any society.

May (1952) introduced three axioms N, A, and PR for social welfare functions. The first two axioms N and A are very natural: Condition N states that a welfare function should be indifferent to the alternatives. Condition A states that the aggregate preference should be completely independent of the exact numbering of the voters.

Neutrality (N). For any profile $R \in \{-1, 0, 1\}^n$, we have $F(-R) = -F(R)$.
Anonymity (A). For any profiles R, $R' \in \{-1, 0, 1\}^n$ where the preferences in R' are a permutation of the preferences in R', we have $F(R) = F(R')$.

May's third axiom PR, however, is usually criticized as being 'too strong'. Campbell and Kelly (2000) even state that ''*it is not at all clear why it should be imposed*''. PR requires that whenever the aggregate preference is indifferent, and a single voter changes his mind, then the new aggregate preference must follow this single voter.

Positive Responsiveness (PR). For any profiles R, $R' \in \{-1, 0, 1\}^n$ with $R'_i \geq R_i$ for all $i \in S$ and $R'_j > R_j$ for some $j \in S$, we have that $F(R) \geq 0$ implies $F(R') = 1$. For any profiles R, $R' \in \{-1, 0, 1\}^n$ with $R'_i \leq R_i$ for all $i \in S$ and $R'_j < R_j$ for some $j \in S$, we have that $F(R) \leq 0$ implies $F(R') = -1$.

Other natural axioms for social welfare functions from the literature are Pareto Optimality (PO) and Weak Pareto Optimality (WPO). PO states that if one alternative is preferred by none of the voters, whereas the other alternative is preferred by some voter, then this other preference should become the aggregate preference. WPO states that if all voters agree, then also the aggregate preference must agree with them. Clearly, PO implies WPO.

Pareto Optimality (PO). For any profile $R \in \{-1, 0, 1\}^n$ with $R_i \geq 0$ for all $i \in S$ and $R_j = 1$ for some $j \in S$, we have $F(R) = 1$. For any profile $R \in \{-1, 0, 1\}^n$ with $R_i \leq 0$ for all $i \in S$ and $R_j = -1$ for some $j \in S$, we have $F(R) = -1$.
Weak Pareto Optimality (WPO). If $R_i = 1$ holds for all $i \in S$, then $F(R) = 1$. If $R_i = -1$ holds for all $i \in S$, then $F(R) = -1$.

As the main contribution of this note, we propose the new axiom '*Reducibility to Subsocieties*' (RS) for social welfare functions. A *subsociety* results from removing one of the voters from society S. RS states that instead of asking the n voters in S, one could equivalently ask the n possible subsocieties of the n voters, and form the aggregate preference of their aggregate preferences. For a profile $R \in \{-1, 0, 1\}^n$ with $n \geq 2$, let $R^{-i} \in \{-1, 0, 1\}^{n-1}$ denote the profile that results from removing the ith voter from profile R.

Reducibility to Subsocieties (RS). For any profile $R \in \{-1, 0, 1\}^n$ with $n \geq 2$, we have $F(R) = F(F(R^{-1}), F(R^{-2}), \ldots, F(R^{-n}))$.

In certain university councils (and in many other organizational councils and committees), when the chairman is monitoring a ballot, then sometimes he abstains from voting himself in order to keep the procedure transparent. Now assume that over time, the chairmanship circulates through the council, and that the voting rule should express the 'average' opinion of all such councils under all the possible chairmen. Then the council without chairman forms a subsociety of the council, just as defined above, and RS stipulates that the aggregate preference of the total council behaves in the same way as the aggregate preference of all the subsocieties under changing chairmen.

When there are exactly $n = 2$ voters, then RS implies anonymity—assuming either N and WPO, or simply that when there is only one individual the aggregate and individual decisions are identical. So, axiom RS embeds *some* kind of anonymity.

The *majority rule* is the social welfare function MAJ that assigns to any profile $R \in \{-1, 0, 1\}^n$ the aggregate preference $\mathrm{MAJ}(R) = \mathrm{sgn}(\Sigma_{i \in S} R_i)$. Here $\mathrm{sgn}(x)$ is the standard sign-function for real numbers x with $\mathrm{sgn}(x) = 1$ for $x > 0$, $\mathrm{sgn}(x) = 0$ for $x = 0$, and $\mathrm{sgn}(x) = -1$ for $x < 0$. A fundamental result of May (1952) is the following characterization of the majority rule.

Proposition 1. (May, 1952) *A social welfare function* $F\colon \bigcup_{n>0} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$ *satisfies N, A, and PR if and only if it is the majority rule.*

Other characterizations of the majority rule are given for instance by Maskin (1995), by Dasgupta and Maskin (1998), by Campbell and Kelly (2000), and by Aşan and Sanver (2002). In this note, we will prove the following new characterization of the majority rule that results from May's characterization by replacing the two axioms A and PR by the two axioms PO and RS.

Theorem 2. *A social welfare function* $F\colon \bigcup_{n>0} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$ *satisfies N, PO, and RS if and only if it is the majority rule.*

Moreover, we will prove that one cannot drop any of the three axioms N, PO, and RS from the statement in Theorem 2 without losing the characterization of the majority rule:

Theorem 3. *There exists a social welfare function* $F\colon \bigcup_{n>0} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$, *that is not the majority rule and that satisfies the axioms*

(a) *N, A, RS, but not PO,*
(b) *N, A, PO, but not RS,*
(c) *A, PO, RS, but not N.*

We will also examine the implications of replacing the strong axiom PO by the weaker axiom WPO in Theorem 2.

Theorem 4. *Let* $F\colon \bigcup_{n>0} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$ *be a social welfare function that satisfies N and RS, and that also satisfies the 'anchor' condition* $F(0, 1) = 1$. *Then F satisfies WPO if and only if F satisfies PO.*

The statements in Theorems 2 and 4 together imply another characterization of the majority rule in terms of N, WPO, RS, and the anchor condition $F(0, 1) = 1$. The anchor condition looks artificial, but it cannot be avoided or weakened, as Theorem 5 below demonstrates.

The *unanimous rule* UNA is the social welfare function that assigns $\mathrm{UNA}(R) = 1$ only if all voters i in R agree and have $R_i = 1$, that assigns $\mathrm{UNA}(R) = -1$ only if all voters have $R_i = -1$, and that is tied with $\mathrm{UNA}(R) = 0$ in all other cases.

Theorem 5. *Let* $F\colon \bigcup_{n>0} \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ *be a social welfare function that satisfies N, WPO, and RS.*

(a) *If* $F(0, 1) = 1$, *then* F *is the majority rule.*
(b) *If* $F(0, 1) = 0$, *then* F *is the unanimous rule.*
(c) *If* $F(0, 1) = -1$, *then* F *belongs to an infinite family of non-monotone social welfare functions.*

The social welfare functions in statement (c) are somewhat strange and non-standard, since they violate the so-called Monotonicity axiom. Monotonicity requires that if some of the voters become more favorable for one alternative whereas the others do not change their opinion, then the aggregate cannot become less favorable for this alternative. The functions in (c), however, satisfy $F(0, 0) = 0 > -1 = F(0, 1)$, and thus are non-monotone and only of academic interest.

The proofs of Theorems 2–5 are given in Section 2.

2. Proofs of the theorems

First let us prove Theorem 2. One direction of the statement is straightforward, since the majority rule clearly satisfies the three axioms N, PO, and RS. For the other direction of the statement in Theorem 2, consider an arbitrary social welfare function F that satisfies axioms N, PO, and RS. We will show by induction on n that then $F = \mathrm{MAJ}$. For $n = 1$, axiom PO implies $F(1) = 1 = \mathrm{MAJ}(1)$ and $F(-1) = -1 = \mathrm{MAJ}(-1)$, and axiom N implies $F(0) = 0 = \mathrm{MAJ}(0)$. Hence, for $R \in \{-1, 0, 1\}^1$ we indeed have $F(R) = \mathrm{MAJ}(R)$.

Now assume that $n \geq 2$ and that we have already proved for all profiles $R' \in \{-1, 0, 1\}^{n-1}$ that $F(R') = \mathrm{MAJ}(R')$. Consider a profile $R \in \{-1, 0, 1\}^n$, and let p, z, m (plus, zero, minus) be non-negative integers, such that p of the voters $i \in S$ have $R_i = 1$, z of the voters $i \in S$ have $R_i = 0$, and m of the voters $i \in S$ have $R_i = -1$. This implies $p + z + m = n$. We distinguish three cases, and we will show that in each of these cases $F(R) = \mathrm{MAJ}(R)$ holds.

- If $p = m$ holds, then by the induction hypothesis $F(R^{-i}) = \mathrm{MAJ}(R^{-i}) = -R_i$ for all voters $i \in S$. Then axiom RS yields $F(R) = F(F(R^{-1}), F(R^{-2}), \ldots, F(R^{-n})) = F(-R)$, whereas axiom N yields that $F(-R) = -F(R)$. Therefore $F(R) = -F(R)$, and this yields $F(R) = 0 = \mathrm{MAJ}(R)$.
- If $p \geq m + 1$ holds, and if we eliminate any individual from R, then we will have $p \geq m$ in all cases, and $p > m$ in at least one case. By the induction hypothesis, the aggregate decision is determined by the majority rule when there are $n - 1$ individuals. Therefore, after eliminating each individual, we will either have 1 as aggregate decision (and that must be true in at least one case), or a tie decision 0. Therefore, RS and PO together imply that $F(R) = 1 = \mathrm{MAJ}(R)$ in this case.
- The case $p \leq m - 1$ is symmetric to the previous case and yields $F(R) = \mathrm{MAJ}(R) = -1$.

Since the three cases cover all possibilities for non-negative integers p, z, m, the inductive proof of Theorem 2 is complete.

Next, we turn to Theorem 3. For statement (a), we observe that the social welfare function $F_1(R) \equiv 0$ satisfies N, A, RS, but not PO. For statement (b), the following social welfare function F_2 satisfies axioms N, A, PO: $F_2(R) = 1$ if $R_j = 1$ for some $j \in S$, and $R_i \geq 0$ for all other voters $i \in S$. $F_2(R) = -1$ if $R_j = -1$ for some $j \in S$, and $R_i \leq 0$ for all other voters $i \in S$. $F_2(R) = 0$ in all other cases. Function F_2 does not satisfy RS, since

$$F_2(1, 1, -1) = 0 \neq 1 = F_2(0, 0, 1) = F_2(F_2(1, -1), F_2(1, -1), F_2(1, 1)).$$

For statement (c), the following social welfare function F_3 satisfies A, PO, RS: $F_3(R) = 0$ if $R_i = 0$ for all $i \in S$. $F_3(R) = 1$ if $R_j = 1$ for some $j \in S$, and $R_i \geq 0$ for all other voters $i \in S$. $F_3(R) = -1$ in all other cases. Function F_3 does not satisfy N, since for $R = (1, -1)$ we have $-F_3(R) \neq F_3(-R)$. This completes the proof of Theorem 3.

For the proof of Theorem 4, consider a social welfare function F with $F(0, 1) = 1$, that satisfies N, RS, and WPO. We prove with induction on n that F also satisfies axiom PO. For $n = 1$, WPO yields that $F(1) = 1$ and $F(-1) = -1$. For $n = 2$, WPO yields that $F(1, 1) = 1$ and $F(-1, -1) = -1$. Moreover, $F(0, 1) = 1$ together with N and RS implies that $F(0, -1) = F(-1, 0) = -1$ and $F(1, 0) = 1$. Hence, the claimed statement holds for $n \leq 2$.

For the inductive step, consider any profile $R \in \{-1, 0, 1\}^n$ with $n \geq 3$ such that $R_i \geq 0$ for all $i \in S$ and $R_j = 1$ for some $j \in S$. Exactly as in the proof of Theorem 2, let p, z, m denote the number of voters in R that vote for 1, 0, -1, respectively. Then $m = 0$, $p \geq 1$, and $z = n - p$. First, we consider the case where $p \geq 2$ holds. Then every subsociety of R has at least one voter who votes for 1, whereas all the other voters in the subsociety vote for 0 or 1. By the induction hypothesis, every subsociety has 1 as aggregate decision. Because of RS, $F(R) = F(R')$ where R' has $(p', z', m') = (n, 0, 0)$, and by WPO we get $F(R') = 1$. Second, we consider the case where $p = 1$ and $z = n - 1 \geq 2$. Then $n - 1$ of the subsocieties have one voter who votes for 1, whereas all the other voters vote for 0. In the remaining subsociety, all voters vote for 0. By the induction hypothesis and by RS, we get that $F(R) = F(R'')$ where R'' has $(p'', z'', m'') = (n - 1, 1, 0)$. The analysis of the first case shows that $F(R'') = 1$, and hence $F(R) = 1$.

In either case, we have shown that $F(R) = 1$ holds as desired. The analysis for a profile $R \in \{-1, 0, 1\}^n$ with $n \geq 3$ such that $R_i \leq 0$ for all $i \in S$ and $R_j = -1$ for some $j \in S$ can be done in a symmetric way. This completes the proof of Theorem 4.

Next, let us prove Theorem 5. Note that statement (a) is an immediate consequence of Theorems 2 and 4.

For statement (b), consider a welfare function F that satisfies N, WPO, RS and $F(0, 1) = 0$. Then N implies $F(0, -1) = 0$ and $F(0, 0) = F(-1, 1) = 0$, and WPO implies $F(1, 1) = 1$ and $F(-1, -1) = -1$. Hence, for all $R \in \{-1, 0, 1\}^n$ with $n \leq 2$ we indeed have $F(R) = \mathrm{UNA}(R)$. An inductive argument now shows that $F \equiv \mathrm{UNA}$ for all $n \geq 3$: Assume that we have proved $F(R) \equiv \mathrm{UNA}(R)$ for all profiles with up to $n - 1$ voters, and consider a profile $R \in \{-1, 0, 1\}^n$. As in the proof of Theorem 2, let p, z, m denote the number of voters that vote for 1, 0, -1, respectively.

- If $p = n$, then WPO yields $F(R) = \mathrm{UNA}(R) = 1$.
- If $p = n - 1$, then RS and the induction hypothesis yield that $F(R)$ is the same as $F(R')$ where R' has $(p', z', m') = (1, n - 1, 0)$. Applying RS and the induction hypothesis another time, we get that $F(R') = F(R'')$ where R'' has $(p'', z'', m'') = (0, n, 0)$. With that, N yields $F(R'') = 0$, and hence $F(R) = \mathrm{UNA}(R)$.

- If $2 \leq p \leq n-2$, then RS and the induction hypothesis imply that $F(R) = F(R')$ where R' has $(p', z', m') = (0, n, 0)$. Therefore, $F(R) = \text{UNA}(R) = 0$ in that case.
- The cases with $m \geq 2$ can be handled symmetrically, and all lead to $F(R) = \text{UNA}(R)$.
- Finally, if $p \leq 1$ and $m \leq 1$, then $z \geq n-2 \geq 1$. Then by the induction hypothesis, all subsocieties of R have aggregate vote 0. Then RS and N imply $F(R) = \text{UNA}(R) = 0$.

This completes the argument for statement (b).

For statement (c), consider a welfare function F that satisfies N, WPO, RS and $F(0, 1) = -1$. Then N implies $F(0, -1) = 1$ and $F(0, 0) = F(-1, 1) = 0$, and WPO implies $F(1, 1) = 1$ and $F(-1, -1) = -1$. This fixes all values of F for profiles with $n \leq 2$ voters. One possible extension of F to a function F^* on profiles with $n \geq 3$ voters is the following. Once again, we use p, z, m to denote the number of voters in a profile $R \in \{-1, 0, 1\}^n$ that vote for 1, 0, -1, respectively. Note that $p + z + m = n \geq 3$.

- If $p = n$, then $F^*(R) = 1$. If $m = n$, then $F^*(R) = -1$. If $z = n$, then $F^*(R) = 0$.
- If $\max\{p, z, m\} \leq n-2$, then $F^*(R) = 0$.
- For odd n: If (p, z, m) equals $(0, 1, n-1)$, or $(1, n-1, 0)$, or $(n-1, 0, 1)$, then $F^*(R) = 1$. If (p, z, m) equals $(n-1, 1, 0)$, or $(0, n-1, 1)$, or $(1, 0, n-1)$, then $F^*(R) = -1$.
- For even n: If (p, z, m) equals $(n-1, 0, 1)$, or $(0, n-1, 1)$, or $(0, 1, n-1)$, then $F^*(R) = 1$. If (p, z, m) equals $(1, 0, n-1)$, or $(1, n-1, 0)$, or $(n-1, 1, 0)$, then $F^*(R) = -1$.

Some straightforward but tedious case considerations show that this function F^* satisfies N, RS, and WPO for all $n \geq 3$.

In fact, there is an infinite number of functions satisfying the conditions of Theorem 5(c): For some fixed integer $k \geq 2$, consider the function F_k that agrees with the above function F^* on all profiles R with $n \leq k$ voters, and that agrees with the unanimous rule UNA on all profiles R with $n \geq k+1$ voters. It can be seen that every such function F_k satisfies N, WPO, RS, and $F(0, 1) = -1$. This completes the proof of Theorem 5.

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