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RESEARCH ARTICLE

A one-shot proof of Arrow's impossibility theorem

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Abstract We offer a new proof of the well-known Arrow's impossibility theorem. The proof is simple, very short and it follows from the assumptions in a transparent way.

Keywords Arrow's impossibility theorem · Social welfare function · Dictatorship · (i, j)-pivotal voter

JEL Classification D7 · D70 · D71

1 Introduction

Arrow's impossibility theorem, perhaps one of the most important theorems in economics, has inspired numerous impossibility results, pioneered the field of social choice theory, and attracted scores of different proofs. To demonstrate dictatorship, most proofs follow one of two methods. In the first method, one can prove the theorem by shrinking the decisive voter set to one voter through reverse induction,

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sequentially excluding voters that have no say in the social preferences. This is the original method in Arrow (1951). The second method first identifies a candidate dictator, the so-called pivotal voter, who can alter the social preferences in some way, and then establishes the pivotal voter's role as a dictator over social preferences. This is the method of Barberá (1980), which was improved upon in Geanakoplos (2005) by the use of the extreme pivotal voter and the trick of ordering voters and flipping alternatives.

Our proof attempts to improve on Barberá (1980) and Geanakoplos (2005). We first define the (i, j)-pivotal voter and then show that if she ranks j above any other k, the social welfare function has to do the same, i.e., she dictates over (j, k). This immediately implies the uniqueness of the pivotal voter, so this voter dictates every ordered pair. The third proof in Geanakoplos (2005) also adopts this method of finding the dictator, but our three-step proof manages to reduce the number of three-alternative manipulations to one. This improvement is no coincidence, for the theorem suggests no special role of the extreme positions, while the assumption of independence of irrelevant alternatives leads naturally to the consideration of pairs of alternatives. Another merit of this proof lies in not requiring strict preferences in individual or social rankings.

2 The theorem and the proof

Theorem Individuals numbered $1, 2, \ldots, N$ each have complete, reflexive, and transitive preferences over $M \geq 3$ alternatives $A = \{a_1, \ldots, a_M\}$. The set of *preference profiles* $\mathbb{P}$ is unrestricted with a typical element denoted as an ordered list $\vec{\succeq} = (\succeq_1, \ldots, \succeq_N)$. A *social welfare function* R assigns to each $\vec{\succeq}$ complete, reflexive, and transitive social preferences $\trianglerighteq$ over A, i.e., $R : \mathbb{P} \to \mathcal{P}$, where $\mathcal{P}$ denotes the set of all possible social preferences. Arrow's theorem asserts that it is impossible to construct an R with the following three properties.

(*Unanimity*) For arbitrary alternatives a_i and a_j, if $a_i \succ_n a_j$ (meaning $a_i \succeq_n a_j$ and not $a_j \succeq_n a_i$) for each individual n in $\vec{\succeq}$, then $a_i \vartriangleright a_j$ (meaning $a_i \trianglerighteq a_j$ and not $a_j \trianglerighteq a_i$).

(*AIIA: Arrow's Independence of Irrelevant Alternatives*) If each individual's preferences over a_i and a_j are the same in $\vec{\succeq}$ and $\vec{\succeq}'$, then $R(\vec{\succeq})$ and $R(\vec{\succeq}')$ rank the two alternatives the same.

(*Non-dictatorship*) There exists no individual n such that for each $\vec{\succeq} \in \mathbb{P}$ and its corresponding $\trianglerighteq = R(\vec{\succeq})$, $a_i \succ_n a_j$ always implies $a_i \vartriangleright a_j$.

Proof Suppose R satisfies Unanimity and AIIA. Consider an arbitrary $\vec{\succeq}$ in which $a_i \succ_n a_j$ for all n, and then swap the position of the two alternatives sequentially from 1 to N. According to Unanimity, we start with $a_i \vartriangleright a_j$ and end with $a_j \vartriangleright a_i$. We call the first voter whose swap invalidates $a_i \vartriangleright a_j$ the (i, j)-*pivotal voter* and denote her number n_{ij}. AIIA makes sure that this definition is independent of $\vec{\succeq}$.

	1	$\cdots$	$n_{ij}-1$	n_{ij}	$n_{ij}+1$	$\cdots$	N
$\vec{\succsim}'$	a_j	$\cdots$	a_j	a_i	a_i	$\cdots$	a_i
	a_k	$\cdots$	a_k				
	a_i	$\cdots$	a_i	a_j	a_j	$\cdots$	a_j
				a_k	a_k	$\cdots$	a_k
$\vec{\succsim}''$	$\square$	$\cdots$	$\square$				
	$\square a_j$	$\cdots$	$\square a_j$	a_j	a_i	$\cdots$	a_i
	$\square$	$\cdots$	$\square$		$\square$	$\cdots$	$\square$
	a_i	$\cdots$	a_i	a_i	$\square a_j$	$\cdots$	$\square a_j$
				a_k	$\square$	$\cdots$	$\square$

Consider any $\vec{\succsim}'$ with the depicted rankings of the three alternatives. We must have $a_i \rhd a_j \rhd a_k$, where the first relation is by the definition of n_{ij} and the second by Unanimity. For $\vec{\succsim}''$, squares denote possible positions of a_k, with indifference drawn by putting alternatives at the same level. We have $a_j \trianglerighteq a_i \rhd a_k$, where the first is by the definition of n_{ij} and the second by AIIA (individual preferences over a_i and a_k are the same in $\vec{\succsim}'$ and $\vec{\succsim}''$). Focusing on a_j and a_k, we conclude by AIIA that n_{ij} dictates $a_j \rhd a_k$, i.e.,

$$a_j \succ_{n_{ij}} a_k \text{ implies } a_j \rhd a_k \quad \text{for all } i \neq j \neq k. \tag{$*$}$$

In the swapping process that defines n_{jk}, $(*)$ says that $a_j \rhd a_k$ should not change as long as n_{ij} ranks j above k, so $n_{jk} \geq$ [1] n_{ij}. For n_{kj}, the ranking of the two alternatives should become $a_j \rhd a_k$ no later than n_{ij} makes the change, so $n_{kj} \leq n_{ij}$. We have $n_{jk} \geq n_{ij} \geq n_{kj}$. As j and k are distinct and arbitrary, $n_{kj} \geq n_{jk}$ also holds, implying $n_{jk} = n_{kj} = n_{ij}$, which can be easily extended to all the other n_{ts}'s. But $(*)$ requires that this unique pivotal voter holds dictatorship over all ordered pair of alternatives, violating Non-dictatorship. $\square$

References

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[1] If there are at least four alternatives, this inequality alone can lead to the conclusion. Hint: consider the greatest n_{ij}.