Handout 1: Arrow's Theorem and Related Results

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August 31, 2022

Relations

Let X be a set. A relation B on X is a subset $B \subseteq X \times X$. Recall some standard properties that a relation may satisfy:

1. Reflexive: for all $x \in X$, $x\ B\ x$
2. Irreflexive: for all $x \in X$, not $x\ B\ x$
3. Complete: for all $x, y \in X$, either $x\ B\ y$ or $y\ B\ x$
4. Symmetric: for all $x, y \in X$, $x\ B\ y$ implies $y\ B\ x$
5. Asymmetric: for all $x, y \in X$, $x\ B\ y$ implies not $y\ B\ x$
6. Transitive: for all $x, y, z \in X$, $x\ B\ y$ and $y\ B\ z$ implies $x\ B\ z$

Strict preferences

A binary relation P on a set X is a *strict weak order* if and only if P is asymmetric and satisfies the following condition for all $x, y, z \in X$:

negative transitivity: if xPy, then xPz or zPy.

Negative transitivity is equivalent to the condition that if *not* xPz and *not* zPy, then *not* xPy, which explains the name.

Exercise 1. If P is negative transitive and asymmetric, then P is transitive.

Remark 2. If not $x\ P\ y$ and not $y\ P\ x$, then we say that the voter is *indifferent* between x and y. Note that this blurs the distinction between being *indifferent* and *non-comparable*.

The relation P is a *strict linear order* if and only if it satisfies asymmetry, negative transitivity, and weak completeness: for all $x, y \in X$, if $x \neq y$, then xPy or yPx.

Weak preferences

Some authors take an agents weak preference as primitive. Let $R \subseteq X \times X$, we defined the following relations from R: $P(R)$ (strict preference), $I(R)$ (indifference) and $N(R)$ (non-comparability):

- $xP(R)y$ if, and only if, xRy and not yRx.
- $xI(R)y$ if, and only if, xRy and yRx.
- $xN(R)y$ if, and only if, neither xRy nor yRx.

A minimal assumption is that R is reflexive (so that indifference is reflexive). We say that R is *quasi-transitive* if $P(R)$ is transitive.

Profiles

For any nonempty set X, whose members we call *alternatives*, and any nonempty set V, whose members we call *voters*, a (X,V)-*profile* $\mathbf{P}$ is an element of $\mathcal{O}(X)^V$, i.e., a function assigning to each $i \in V$ a relation $\mathbf{P}(i) \in \mathcal{O}(X)$, which we call *i's strict preference relation*. For $x, y \in X$, let:

$$\begin{aligned} \mathbf{P}(x,y) &= \{i \in V \mid x \, \mathbf{P}_i \, y\}; \\ \mathbf{P}_{|\{x,y\}} &= \text{the function assigning to each } i \in V \text{ the relation } \mathbf{P}_i \cap \{x,y\}^2. \end{aligned}$$

Collective Choice Rules

A (V,X)-*collective choice rule* ((V,X)-CCR) is a function f from a subset of $\mathcal{O}(X)^V$ to $\mathcal{B}(X)$. By $x \; f(\mathbf{P}) \; y$, we mean $(x,y) \in f(\mathbf{P})$.

Constant function

Let $P \in \mathcal{L}(X)$. Then define f_P as follows: for all $\mathbf{P}$, $f_P(\mathbf{P}) = P$. Let P be a strict linear order with $cPbPa$ and $\mathbf{P}$ the following profile.

1	1	1
a	a	a
b	b	b
c	c	c

$b f_P(\mathbf{P}) a$ even though $\mathbf{P}(a,b) = V$.

Majority ordering

Define f_{maj} as follows: for all $\mathbf{P}$, $f_{maj}(\mathbf{P}) = \{(x,y) \mid |\mathbf{P}(x,y)| > |\mathbf{P}(y,x)|\}$
(so, $f_{maj}(\mathbf{P})$ is the majority ordering). Let $\mathbf{P}$ be the following profile (a Condorcet cycle):

1	1	1
a	b	c
b	c	a
c	a	b

$f_{maj}(\mathbf{P})$ is not transitive.

Majority ordering with limited domain

Define f^*_{maj} as follows: For each $\mathbf{P}$ such that there are no cycles in $\mathcal{M}(\mathbf{P})$, $f^*_{maj}(\mathbf{P})$ is the majority ordering for $\mathbf{P}$.
Then, $dom(f^*_{maj}) \subsetneq \mathcal{O}(X)^V$.

Pareto/Unanimity

Define f_u as follows: for all $\mathbf{P}$, $f_u = \{(x,y) \mid \mathbf{P}(x,y) = V\}$. Let $\mathbf{P}$ be the following profile:

1	1	1
a	b	b
b	c	a
c	a	c

$b\ f_u(\mathbf{P})\ c$ and not $a\ f_u(\mathbf{P})\ c$, but not $b\ f_u(\mathbf{P})\ a$.

Thus, f_u generates rankings that violate negative transitivity.

Notation Given a strict linear order P on X, the *rank* of x in P is $rank(x, P) = |\{x \mid x \in X,\ xPy\}| + 1$. Define the following scoring functions for an alternative $x \in X$ and strict linear order P on X:

$$score_{PL}(x, P) = \begin{cases} 1 & \text{if } rank(x, P) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$score_B(x, P) = m - rank(x, P)$, where $m = |X|$.

Then, given a profile $\mathbf{P}$ and $x \in X$,

$$\begin{array}{rrcl} \text{(Plurality score)} & PL_{\mathbf{P}}(x) & = & \sum_{i \in V} score_{PL}(x, \mathbf{P}_i) \\ \text{(Borda score)} & BS_{\mathbf{P}}(x) & = & \sum_{i \in V} score_B(x, \mathbf{P}_i) \end{array}$$

Plurality ranking

Define f_{pl} as follows: for all $\mathbf{P} \in \mathcal{L}(X)^V$, $f_{pl}(\mathbf{P}) = \{(x, y) \mid PL_{\mathbf{P}}(x) \geq PL_{\mathbf{P}}(y)\}$.

$\mathbf{P}$:

2	1
a	b
c	c
d	d
b	a

not $c\ f_{pl}(\mathbf{P})\ d$ even though $\mathbf{P}(c, d) = V$.

$\mathbf{P}$:

40	35	25
t	c	k
k	k	t
c	t	c

$\mathbf{P}'$:

40	35	25
t	k	k
k	t	t
c	c	c

$t\ f_{pl}(\mathbf{P})\ c\ f_{pl}(\mathbf{P})\ k$
$k\ f_{pl}(\mathbf{P}')\ t\ f_{pl}(\mathbf{P}')\ c$
$\mathbf{P}_{|\{k,t\}} = \mathbf{P}'_{|\{k,t\}}$

Borda ranking

Define f_{borda} as follows: for all $\mathbf{P} \in \mathcal{L}(X)^V$, $f_{borda}(\mathbf{P}) = \{(x, y) \mid BS_{\mathbf{P}}(x) \geq BS_{\mathbf{P}}(y)\}$.

$\mathbf{P}$:

45	55
a	b
c	a
b	c

$\mathbf{P}'$:

45	55
a	b
b	a
c	c

$a\ f_{borda}(\mathbf{P})\ b\ f_{borda}(\mathbf{P})\ c$
$b\ f_{borda}(\mathbf{P}')\ a\ f_{borda}(\mathbf{P}')\ c$
$\mathbf{P}_{|\{a,b\}} = \mathbf{P}'_{|\{a,b\}}$

Dropping anonymity and neutrality

Let $T \subseteq X \times X$ be a strict linear order on X and $R \subseteq V \times V$ be a strict linear order on V. For each $\mathbf{P} \in \mathcal{O}(X)^V$ and $x, y \in X$, let $x\ f_{T,R}(\mathbf{P})\ y$ if, and only if, either

1. $x\mathbf{P}_i y$ and there is no j such that jRi and $y\mathbf{P}_j x$; or
2. xTy and there is no $i \in V$ such that $x\mathbf{P}_i y$ or $y\mathbf{P}_i x$

Axioms

Domain Conditions

- universal domain (UD): $\mathrm{dom}(f) = \mathcal{O}(X)^V$.
- linear domain (LD): $\mathrm{dom}(f) = \mathcal{L}(X)^V$.

Codomain Conditions ("Rationality Postulates")

- transitive rationality (TR): for all $\mathbf{P} \in \mathrm{dom}(f)$, $f(\mathbf{P})$ is transitive.
- full rationality (FR): for all $\mathbf{P} \in \mathrm{dom}(f)$, $f(\mathbf{P})$ is a strict weak order.

Interprofile Conditions

- independence of irrelevant alternatives (IIA): for all $\mathbf{P}, \mathbf{P}' \in \mathrm{dom}(f)$ and $x, y \in X$, if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P}')y$.

Decisiveness Conditions

- Pareto (P): for all $\mathbf{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if $\mathbf{P}(x, y) = V$, then $xf(\mathbf{P})y$.
- dictatorship: there is an $i \in V$ such that for all $\mathbf{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if xP_iy, then $xf(\mathbf{P})y$.

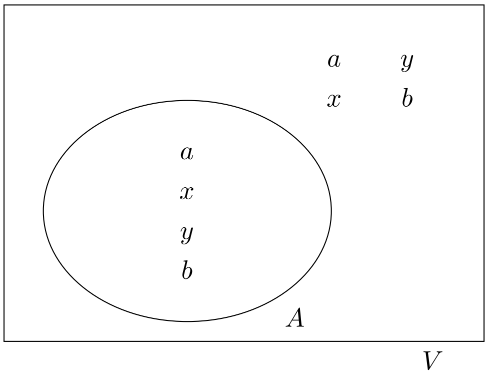
Theorem 3 (Arrow)**.** Assume that $|X| \geq 3$ and V is finite. Then any (V, X)-CCR for satisfying UD, IIA, FR, and P is a dictatorship.

Decisive Coalitions

Definition 4 (Decisive Coalition)**.** $A \subseteq V$ is decisive for x over y, if for all $\mathbf{P} \in \mathit{dom}(f)$, if $A \subseteq \mathbf{P}(x, y)$, then $x\ f(\mathbf{P})\ y$.

Lemma 5 (Decisiveness Spread)**.** For any $A \subseteq V$ and candidates $x, y \in X$, if A is decisive for x over y, then for any $z, w \in X$, A is decisive for z over w.

Proof. Let $\mathbf{P} \in \mathit{dom}(f)$ be a profile with $A \subseteq \mathbf{P}(a, b)$. We must show that $a\ f(\mathbf{P})\ b$. Let $\mathbf{P}'$ be the profile in which $\mathbf{P}'_{|\{a,b\}} = \mathbf{P}_{|\{a,b\}}$ and the rankings of a, b, x, and y are as follows:



1. Pareto implies that $a\ f(\mathbf{P}')\ x$
2. Pareto implies that $y\ f(\mathbf{P}')\ b$
3. A is decisive for x over y implies that $x\ f(\mathbf{P}')\ y$
4. $f(\mathbf{P}')$ is (quasi-)transitive, so:
   (a) 1. and 3. implies that $a\ f(\mathbf{P}')\ y$
   (b) 4(a). and 2. implies that $a\ f(\mathbf{P}')\ b$

Since $\mathbf{P}'_{|\{a,b\}} = \mathbf{P}_{|\{a,b\}}$ and $a\ f(\mathbf{P}')\ b$, by IIA, $a\ f(\mathbf{P})\ b$. (Note that the above proof works even if $x = y$.) $\square$

Lemma 5 means that if A is decisive for some x over y, then A is decisive for *all* x over y. We say that A is decisive if A is decisive for some (hence all) x over y. Let $\mathcal{D} = \{A \mid A \text{ is decisive}\}$. Then:

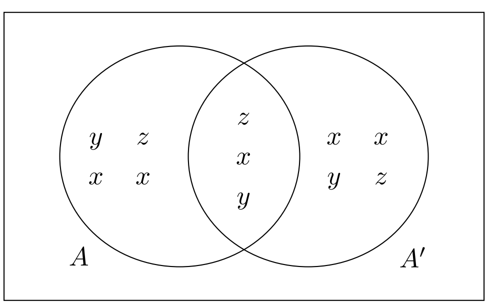
- By Pareto, $V \in \mathcal{D}$. So, $\mathcal{D} \neq \emptyset$
- Since V is finite, there is a minimal $A \in \mathcal{D}$: an $A \in \mathcal{D}$ such that there is no $B \in \mathcal{D}$ with $B \subsetneq A$.

Lemma 6. Suppose that f is a CCR for X, V and $\mathcal{D}$ is the set of all decisive sets for f. If $A, A' \in \mathcal{D}$ are minimal, then $A = A'$.

Proof. Suppose that f is a CCR for X, V and $\mathcal{D}$ is the set of all decisive sets for f with $A, A' \in \mathcal{D}$. We show that (i) $A \cap A' \neq \emptyset$ and (ii) $A \cap A'$ is decisive for f.

To show (i), suppose that $A \cap A' = \emptyset$. Let $\mathbf{P}$ be a profile in which $A \subseteq \mathbf{P}(x, y)$ and $A' \subseteq \mathbf{P}(y, x)$. Since A is decisive, $x\ f(\mathbf{P})\ y$ and since A' is decisive, $y\ f(\mathbf{P})\ x$. This is a contradiction since $f(\mathbf{P})$ is asymmetric.

To show (ii), let $z \neq x$ and $z \neq y$. Suppose that $\mathbf{P}$ is a profile in which $A \cap A' \subseteq \mathbf{P}(z, y)$. Suppose that $\mathbf{P}'$ is a profile with $\mathbf{P}'_{|\{z,y\}} = \mathbf{P}_{|\{z,y\}}$, and the rankings of x, y and z are as follows:



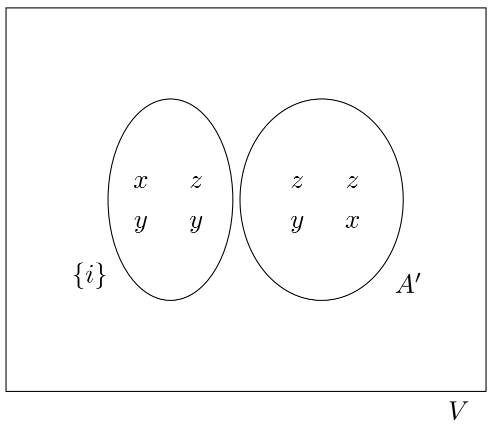
1. $A \in \mathcal{D}$ and $A \subseteq \mathbf{P}'(z, x)$ implies that $z\ f(\mathbf{P}')\ x$
2. $A' \in \mathcal{D}$ and $A' \subseteq \mathbf{P}'(x, y)$ implies that $x\ f(\mathbf{P}')\ y$
3. $f(\mathbf{P}')$ is (quasi-)transitive, so:
   (a) 1. and 2. implies that $z\ f(\mathbf{P}')\ y$

Since $\mathbf{P}'_{|\{z,y\}} = \mathbf{P}_{|\{z,y\}}$ and $z\ f(\mathbf{P}')\ y$, by IIA, $z\ f(\mathbf{P})\ y$. So $A \cap A'$ is decisive for z over y and by Lemma 5, $A \cap A' \in \mathcal{D}$. The contradicts the assumption that A and A' are minimal. $\square$

Hence, there is a unique minimal element of $\mathcal{D}$.

Lemma 7. Let A^* be the unique minimal element of $\mathcal{D}$. For all $i \in A^*$ and $\mathbf{P} \in dom(f)$, if $x\mathbf{P}_i y$, then not $y\ f(\mathbf{P})\ x$.

Proof. Suppose not. There is a $\mathbf{P} \in dom(f)$ and $i \in A^*$ such that $x\mathbf{P}_i y$ and $y\ f(\mathbf{P})\ x$. Since A^* is decisive for x over y, there must be some $A' \subsetneq A^*$ such that $A' \neq \emptyset$ and for all $j \in A'$, not $x\mathbf{P}_j y$. (In particular, $i \notin A'$). We show that A' is decisive for z over x. Let $\mathbf{P}'' \in dom(f)$ be a profile with $A' \subseteq \mathbf{P}''(z,x)$. Consider the profile $\mathbf{P}'$ with $\mathbf{P}'_{|\{x,y\}} = \mathbf{P}_{|\{x,y\}}$ and $\mathbf{P}'_{|\{x,z\}} = \mathbf{P}''_{|\{x,z\}}$ and the remaining rankings of x, y and z are as follows:



1. $A' \cup \{i\} = A^* \in \mathcal{D}$ and $A^* \subseteq \mathbf{P}'(z,y)$ implies that $z\ f(\mathbf{P}')\ y$
2. IIA, $\mathbf{P}'_{|\{x,y\}} = \mathbf{P}_{|\{x,y\}}$, and $y\ f(\mathbf{P})\ x$ implies that $y\ f(\mathbf{P}')\ x$.
3. $f(\mathbf{P}')$ is (quasi-)transitive, so:
   (a) 1. and 2. implies that $z\ f(\mathbf{P}')\ x$

Since $\mathbf{P}''_{|\{x,z\}} = \mathbf{P}'_{|\{x,z\}}$ and $z\ f(\mathbf{P}')\ x$, by IIA, $z\ f(\mathbf{P}'')\ x$. So, A' is decisive for z over x. By Lemma 5, $A' \in \mathcal{D}$. This contradicts the minimality of A^*. $\square$

Definition 8 (Oligarchy)**.** Suppose that f is a CCR for (X, V). A set $A \subseteq V$ is an *oligarchy* for f if A is decisive for f and

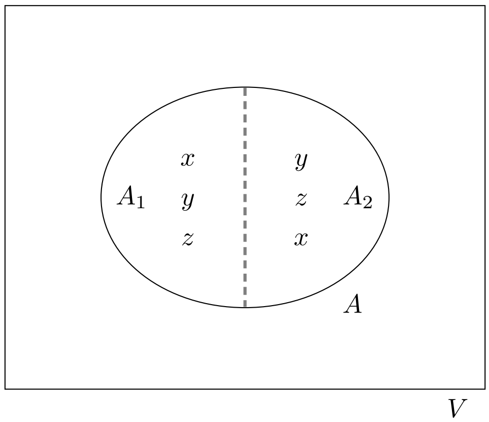
For $\mathbf{P} \in dom(f)$, if $x\mathbf{P}_i y$ for some $i \in A$, then not $y\ f(\mathbf{P})\ x$.

Theorem 9 (Gibbard's Oligarchy Theorem)**.** Assume that $|X| \geq 3$ and V is finite. Then any CCR for $\langle X, V\rangle$ satisfying UD, IIA, TR, and P has an oligarchy.

Proving Arrow's Theorem

Lemma 10. Assume that $|X| \geq 3$ and V is finite. Suppose that f is a CCR for $\langle X, V\rangle$ satisfying UD, IIA, FR, and P. Then, if A is an oligarchy for f, then $|A| = 1$ (so A f is a dictatorship).

Proof. Suppose not. That is, there is an oligarchy A for f with $|A| > 1$. Then, $A = A_1 \cup A_2$ where $A_1 \cap A_2 = \emptyset$, $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Let $\mathbf{P}$ be a profile with the following rankings of x, y, and z:



1. A is an oligarchy, there is $i \in A_1 \subseteq A$ with $x\mathbf{P}_i y$ implies that not $y\ f(\mathbf{P})\ x$
2. A is an oligarchy, there is $j \in A_2 \subseteq A$ with $z\mathbf{P}_j x$ implies that not $x\ f(\mathbf{P})\ z$
3. $f(\mathbf{P}')$ is negatively transitive, so:
   (a) 1. and 2. implies that not $y\ f(\mathbf{P})\ z$
4. $A_1 \cup A_2 = A$ is decisive and $A \subseteq \mathbf{P}(y, z)$ implies that $y\ f(\mathbf{P})\ z$. This contradicts 3.

□

Non-Paretian Social Choice

Let $\mathcal{O}(X)$ be the set of all complete, reflexive and transitive relations and $\mathcal{B}(X)$ the set of binary relations on X. A CCR is a function $f : \mathcal{D} \to \mathcal{B}(X)$ for some set $\mathcal{D}$ of profiles.

- f is a SWF (social welfare function) if for all $\mathbf{R} \in dom(f)$, $f(\mathbf{R})$ is complete and transitive.
- f is *null* if for all $x, y \in X$ and all $\mathbf{R} \in dom(f)$, not $x\ P(f(\mathbf{R}))\ y$.
- f is *anit-Paretian* if for all $\mathbf{R} \in dom(f)$, if $xP(\mathbf{R}_i)y$ for all $i \in V$, then $y\ P(f(\mathbf{R}))\ x$.
- f is *dis-Paretian* if for all $\mathbf{R} \in dom(f)$, if $xP(\mathbf{R}_i)y$ for all $i \in V$, then $x\ N(f(\mathbf{R}))\ y$.
- non-nullness (NN): f is not null.
- non-imposition (NI): for all $x, y \in X$, there is an $\mathbf{R} \in dom(f)$ such that $x\ f(\mathbf{R})\ y$.
- strict non-imposition (SNI): for all $x, y \in X$ with $x \neq y$, there is an $\mathbf{R} \in dom(f)$ such that $x\ P(f(\mathbf{R}))\ y$.
- inverse-dictator: d is an *inverse dictator* if for all $\mathbf{R} \in \mathrm{dom}(f)$ and $x, y \in X$, if $xP(\mathbf{R}_i)y$, then $y\ P(f(\mathbf{R}))\ x$.

Theorem 11 (Murakami 1968)**.** Any SWF satisfying UD, IIA and SNI is either Paretian or anti-Paretian.

Theorem 12 (Murakami 1968)**.** Any SWF satisfying UD, IIA and SNI is either dictatorial or inversely dictatorial.

Theorem 13 (Malawski and Zhou 1994)**.** Any SWF satisfying UD, IIA, and NI is either null, Paretian, or anti-Paretian.

Theorem 14 (Wilson 1972)**.** Any SWF satisfying satisfying UD, IIA, and NI is either null, dictatorial, or inversely dictatorial.

Theorem 15 (Holliday and Kelley 2020)**.** Any transitive CCR satisfying UD, IIA, NI, and NN is either Paretian, anti-Paretian, or dis-Paretian.