New Perspectives on Social Choice

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Lecture 1

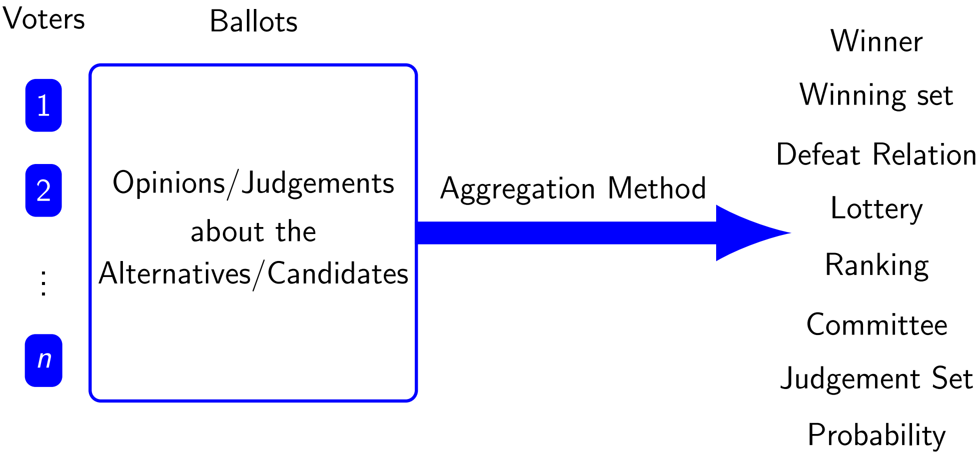
PHIL 808K

Social Choice Theory

"Social choice theory is the study of collective decision processes and procedures. It is not a single theory, but a cluster of models and results concerning the aggregation of individual inputs (e.g., votes, preferences, judgments, welfare) into collective outputs (e.g., collective decisions, preferences, judgments, welfare)."

C. List. *Social Choice Theory*. Stanford Encyclopedia of Philosophy, 2013.

Social Choice Theory



Social Choice Theory

Voters

1

2

⋮

n

Ballots

Opinions/Judgements about the Alternatives/Candidates

Aggregation Method

Winner

Winning set

Defeat Relation

Lottery

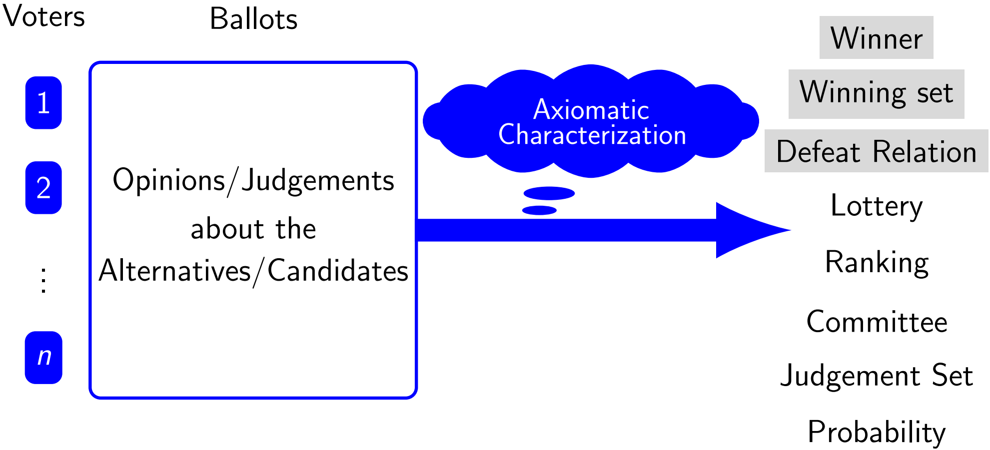
Ranking

Committee

Judgement Set

Probability

Social Choice Theory



Profiles

Fix infinite sets $\mathcal{V}$ and $\mathcal{X}$ of *voters* and *candidates*, respectively.

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Definition

Given nonempty finite $V \subseteq \mathcal{V}$ and $X \subseteq \mathcal{X}$, a (V, X)-**profile** is a function $\boldsymbol{P}$ assigning to each $i \in V$ a binary relation on X.

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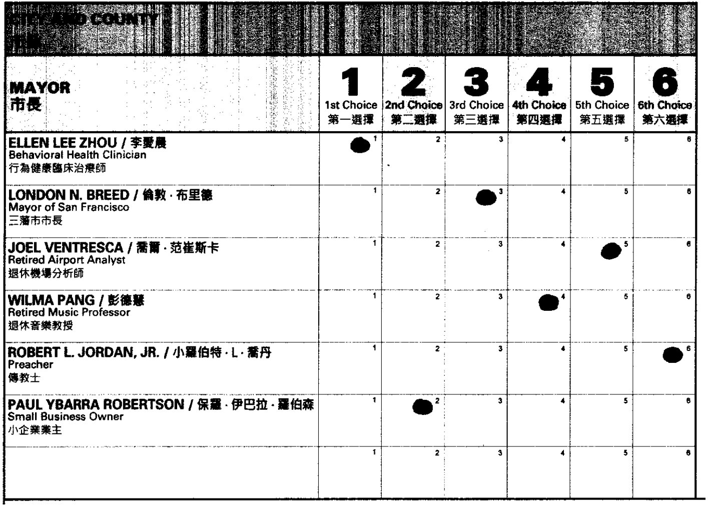
Given nonempty finite $V \subseteq \mathcal{V}$ and $X \subseteq \mathcal{X}$, a (V, X)-**profile** is a function $\boldsymbol{P}$ assigning to each $i \in V$ a binary relation on X.

We write $V(\boldsymbol{P})$ for $\boldsymbol{P}$'s set of voters and $X(\boldsymbol{P})$ for $\boldsymbol{P}$'s set of candidates.

We call $\boldsymbol{P}(i)$ voter i's *ranking*, and we write '$x\boldsymbol{P}_i y$' for $(x, y) \in \boldsymbol{P}(i)$. We take $x\boldsymbol{P}_i y$ to mean that voter i strictly prefers candidate x to candidate y.

Rankings



CITY AND COUNTY						
MAYOR 市長	1 1st Choice 第一選擇	2 2nd Choice 第二選擇	3 3rd Choice 第三選擇	4 4th Choice 第四選擇	5 5th Choice 第五選擇	6 6th Choice 第六選擇
ELLEN LEE ZHOU / 李愛晨 Behavioral Health Clinician 行為健康臨床治療師	1	2	3	4	5	6
LONDON N. BREED / 倫敦 · 布里德 Mayor of San Francisco 三藩市市長	1	2	3	4	5	6
JOEL VENTRESCA / 喬爾 · 范崔斯卡 Retired Airport Analyst 退休機場分析師	1	2	3	4	5	6
WILMA PANG / 彭德懸 Retired Music Professor 退休音樂教授	1	2	3	4	5	6
ROBERT L. JORDAN, JR. / 小羅伯特 · L · 喬丹 Preacher 傳教士	1	2	3	4	5	6
PAUL YBARRA ROBERTSON / 保羅 · 伊巴拉 · 羅伯森 Small Business Owner 小企業業主	1	2	3	4	5	6
	1	2	3	4	5	6

Preferences (Rankings)

Suppose that $B \subseteq X \times X$ is a binary relation.

asymmetry: if xBy, then *not* yBx;

negative transitivity: if xBy, then xBz or zBy.

Negative transitivity

if xBy, then xBz or zBy

Negative transitivity is equivalent to the condition that if *not* xBz and *not* zBy, then *not* xBy, which explains the name.

Negative transitivity

$$\text{if } xBy\text{, then } xBz \text{ or } zBy$$

Negative transitivity is equivalent to the condition that if *not* xBz and *not* zBy, then *not* xBy, which explains the name.

Together negative transitivity and asymmetry imply that B is transitive:

transitivity: if xBy and yBz, then xBz.

B is a *strict weak order* if and only if B satisfies asymmetry and negative transitivity

B is a *strict linear order* if and only if it satisfies asymmetry, transitivity, and weak completeness: for all $x, y \in X$, if $x \neq y$, then xBy or yBx.

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$\mathcal{B}(X)$ is the set of all asymmetric binary relations on X;
$\mathcal{O}(X)$ is the set of all strict weak orders on X;
$\mathcal{L}(X)$ is the set of all strict linear orders on X.

Non-comparability

Let xNy if and only if neither xBy nor yBx. We call N the relation of *non-comparability*.

If B is a strict weak order, then N satisfies the following for all $x, y, z \in X$:

transitivity of non-comparability: if xNy and yNz, then xNz.

(Linear) Profiles

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
b	b	b	b	b	b	b	a	a	a	a	a	a	a	a
c	c	c	c	c	c	c	c	c	c	c	c	b	b	b
a	a	a	a	a	a	a	b	b	b	b	b	c	c	c

(Linear) Anonymous Profile

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
b	b	b	b	b	b	b	a	a	a	a	a	a	a	a
c	c	c	c	c	c	c	c	c	c	c	c	b	b	b
a	a	a	a	a	a	a	b	b	b	b	b	c	c	c

7	5	3
b	a	a
c	c	b
a	b	c

Margin

Let $\boldsymbol{P}$ be a profile and $a, b \in X(\boldsymbol{P})$. Then the **margin of** a **over** b is:

$$Margin_{\boldsymbol{P}}(a, b) = |\{i \in V(\boldsymbol{P}) \mid a\boldsymbol{P}_i b\}| - |\{i \in V(\boldsymbol{P}) \mid b\boldsymbol{P}_i a\}|.$$

40	35	25
t	r	k
k	k	r
r	t	t

$$\begin{aligned}
Margin_{\boldsymbol{P}}(t, k) &= 40 - 60 = -20 \\
Margin_{\boldsymbol{P}}(k, t) &= 60 - 40 = 20 \\
Margin_{\boldsymbol{P}}(k, r) &= 30 \\
Margin_{\boldsymbol{P}}(r, k) &= -30 \\
Margin_{\boldsymbol{P}}(t, r) &= -20 \\
Margin_{\boldsymbol{P}}(r, t) &= 20
\end{aligned}$$

Majority Graph

Let $\boldsymbol{P}$ be a profile and $a, b \in X(\boldsymbol{P})$. Then the **margin of** a **over** b is:

$$Margin_{\boldsymbol{P}}(a,b) = |\{i \in V(\boldsymbol{P}) \mid a\boldsymbol{P}_i b\}| - |\{i \in V(\boldsymbol{P}) \mid b\boldsymbol{P}_i a\}|.$$

We say that a is **majority preferred** to b in $\boldsymbol{P}$ when $Margin_{\boldsymbol{P}}(a,b) > 0$.

40	35	25
t	r	k
k	k	r
r	t	t

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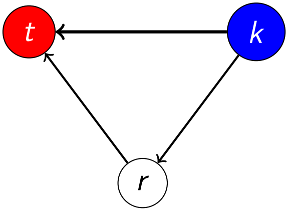


Margin Graph

The **margin graph of** P, $\mathcal{M}(P)$, is the weighted directed graph whose set of nodes is $X(P)$ with an edge from a to b weighted by $Margin(a, b)$ when $Margin(a, b) > 0$. We write

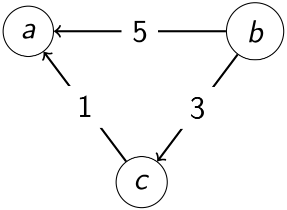
$$a \xrightarrow{\alpha}_{\boldsymbol{P}} b \text{ if } \alpha = Margin_{\boldsymbol{P}}(a, b) > 0.$$

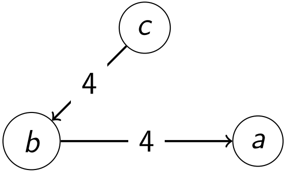
40	35	25
t	r	k
k	k	r
r	t	t

$$\begin{tikzcd} t & & k \arrow[ll, "20"'] \arrow[dl, "30"] \\ & r \arrow[ul, "20"] & \end{tikzcd}$$

Margin Graph

A **margin graph** is a weighted directed graph $\mathcal{M}$ where all the weights have the same parity.





Theorem (Debord, 1987)

For any margin graph $\mathcal{M}$, there is a linear profile $\boldsymbol{P}$ such that $\mathcal{M}$ is the margin graph of $\boldsymbol{P}$.

Collective choice rules

Definition

A (V, X)-**collective choice rule** (or (V, X)-CCR) is a function f such that for any (V, X)-profile $\boldsymbol{P} \in \mathrm{dom}(f)$, $f(\boldsymbol{P})$ is a binary relation on X.

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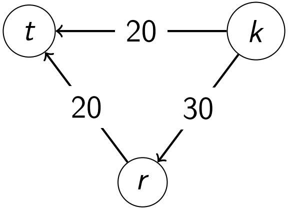
We assume that $f(\boldsymbol{P})$ is at least asymmetric, reflecting our interpretation of $(x, y) \in f(\boldsymbol{P})$ as meaning that x is *strictly socially preferred* to y, or x *defeats* y.

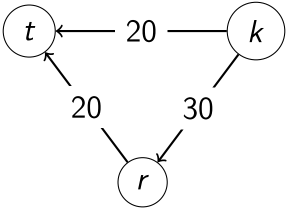
40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k\ f(\boldsymbol{P})\ r\ f(\boldsymbol{P})\ t$

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

k r t

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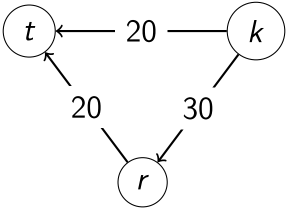

Social Ranking
$k \quad r \quad t$

Majority Ordering, Copeland, Borda

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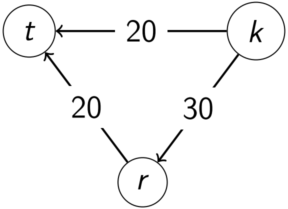

Social Ranking

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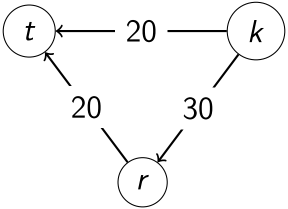
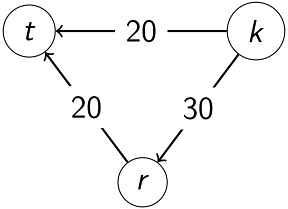


Social Ranking

$k \; r \; t$ Majority Ordering, Copeland, Borda

$k \; t \; r$ Minimax

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k \; r \; t$	Majority Ordering, Copeland, Borda
$k \; t \; r$	Minimax
$r \; k \; t$	

40	35	25
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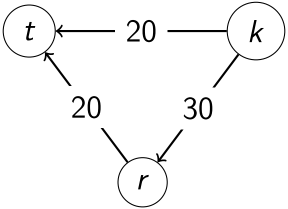

Social Ranking

$k \ r \ t$	Majority Ordering, Copeland, Borda
$k \ t \ r$	Minimax
$r \ k \ t$	Instant Runoff

40	35	25
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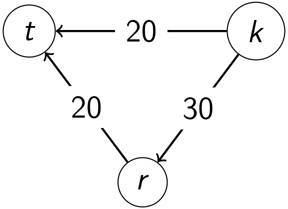

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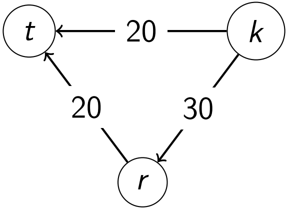
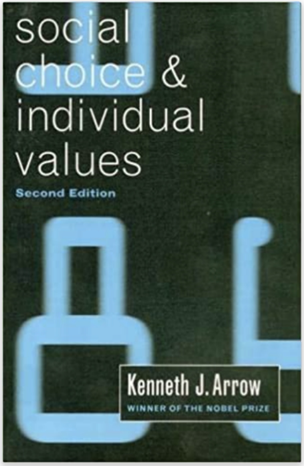


Social Ranking

$k \quad r \quad t$	Majority Ordering, Copeland, Borda
$k \quad t \quad r$	Minimax
$r \quad k \quad t$	Instant Runoff
$t \quad r \quad k$	Plurality

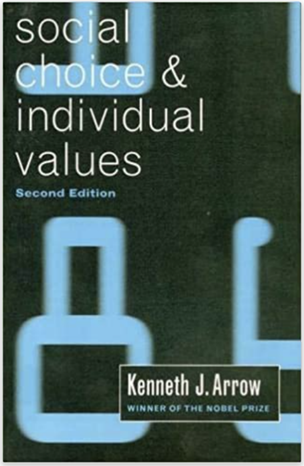
Arrow's Impossibility Theorem



"For an area of study to become a recognized field, or even a recognized subfield, two things are required: It must be seen to have coherence, and it must be seen to have depth. The former often comes gradually, but the latter can arise in a single flash of brilliance....With social choice theory, there is little doubt as to the seminal result that made it a recognized field of study: Arrow's impossibility theorem."

A. Taylor, Social Choice and the Mathematics of Manipulation

Arrow's Impossibility Theorem



K. Arrow. *Social Choice & Individual Values*. Yale University Press (1951).

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P. Suppes. *The pre-history of Kenneth Arrow's social choice and individual values*. Social Choice and Welfare 25(2):319-326 (2015).

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The top three candidates were in a **majority cycle**:

$$
\begin{tikzcd}
 & \text{Flanagan} \arrow[dr, "86"] & \\
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Postulates: Domain conditions

universal domain (UD): $\mathrm{dom}(f) = O(X)^V$.

linear domain (LD): $\mathrm{dom}(f) = L(X)^V$.

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linear domain (LD): $\mathrm{dom}(f) = L(X)^V$.

$f : \mathcal{L}(X)^V \to \mathcal{B}(X)$

If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings. (Arrow 1951 [1963]: 24)

Postulates: Codomain conditions (rationality postulates)

transitive rationality (TR): for all $\boldsymbol{P} \in \mathrm{dom}(f)$, $f(\boldsymbol{P})$ is transitive.

full rationality (FR): for all $\boldsymbol{P} \in \mathrm{dom}(f)$, $f(\boldsymbol{P})$ is a strict weak order.

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The need to construct choices from inputs and to defend those choices is inarguably a necessity of any desirable collective choice procedure. Transitivity is a compelling condition because it provides the strongest possible defense of a collective choice; it ensures that the choice procedure generates a comparison of alternatives akin to a "greater than or equal to" relationship, as used to describe numbers on a line. At the same time, we find transitivity to be the least compelling of Arrow's axioms from the standpoint of constructing a legitimate collective choice procedure. Patty and Penn, Legitimacy and Social Choice

Postulates: Interprofile conditions

independence of irrelevant alternatives (IIA): for all $\boldsymbol{P}, \boldsymbol{P}' \in \mathrm{dom}(f)$ and $x, y \in X$, if $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then $xf(\boldsymbol{P})y$ if and only if $xf(\boldsymbol{P}')y$.

Postulates: Decisiveness conditions

Pareto (P): for all $\boldsymbol{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if for all $i \in V$, $x\ \boldsymbol{P}_i\ y$, then $x\ f(\boldsymbol{P})\ y$.

non-dictatorship: there is no $i \in V$ such that for all $\boldsymbol{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if xP_iy, then $x\ f(\boldsymbol{P})\ y$.

Postulates: Decisiveness conditions

> Taking a broader view for a moment and asking why aggregation is interesting, it is apparent that there must (or should) be some normative claim for the collective ranking to be formed from the included criteria. This is also partially implied by the no dictator axiom: a baseline presumption for embarking upon the design of an aggregation rule is that there is some reason to aggregate (at least some of) the criteria in question....the normative appeal of Pareto is at least partially founded upon the presumption that the inputs are themselves properly coded ("higher" is "better") and, even more important, "well-ordered" in the sense that the inputs are themselves transitive.
>
> (Patty and Penn, Legitimacy and Social Choice, p. 65)

Arrow's Theorem

Theorem (Arrow, 1951) Assume that $|X| \geq 3$ and V is finite. Then any (V, X)-CCR satisfying UD, IIA, FR, and P is a dictatorship.

K. Arrow. *Social Choice and Individual Values*. Yale University Press (1951, 2nd ed., 1963, 3rd ed., 2012).

M. Morreau (2019). *Arrow's Theorem*. Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/entries/arrows-theorem/.

J. S. Kelly (1978). *Arrow Impossibility Theorems*. New York: Academic Press.

Eric Maskin and Amartya Sen (2014). *The Arrow Impossibility Theorem*. (Kenneth J. Arrow Lecture Series), Columbia University Press.

J. Geanakoplos (2005). *Three brief proofs of Arrow's Impossibility Theorem*. Economic Theory, 26, pp. 211 - 215.

Decisive coalitions

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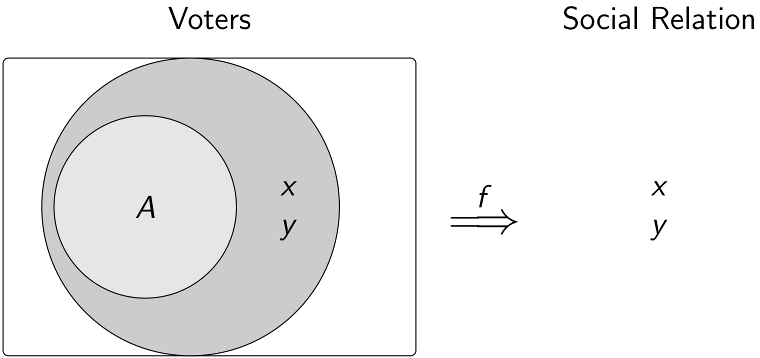
W. Holliday and EP (2020). *Arrow's Decisive Coalitions*. Social Choice and Welfare Social, 54, pp. 463 - 505.

The goal of this paper is a fine-grained analysis of reasoning about decisive coalitions, formalizing how the concept of a decisive coalition gives rise to a social choice theoretic language and logic all of its own.

Decisive coalitions

A coalition $A \subseteq V$ is **decisive for** x **over** y **according to** f if for all (V, X)-profiles $\boldsymbol{P}$, if $x\boldsymbol{P}_i y$ for all $i \in A$, then $xf(\boldsymbol{P})y$.



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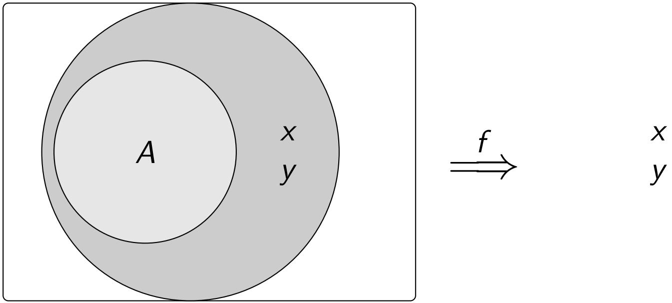
Voters Social Relation



A is **decisive according to** f if for all distinct x, y, A is decisive for x over y.

Decisive Coalitions

- Contagion: If A is decisive for x over y according to f, then A is decisive according to f.
- V is decisive for f, $\varnothing$ is not decisive according to f.
- If A is decisive for f and $A \subseteq B$, then B is decisive according to f.
- If A is decisive according to f and B is decisive according to f, then $A \cap B$ is decisive according to f.
- For any A, either A or $V \setminus A$ is decisive according to f.

Pivotal Voter

J. Geanakoplos (2005). *Three brief proofs of Arrow's Impossibility Theorem*. Economic Theory, 26, pp. 211 - 215.

M. Fey (2014). *A Straightforward Proof of Arrow's Theorem* . Economics Bulletin, AccessEcon, 34(3), pp. 1792 - 1797.

Pivotal Voter

$$\boldsymbol{P} \quad \begin{array}{ccccc} v_1 & v_2 & \cdots & v_{n-1} & v_n \\ \hline a & a & \cdots & a & a \\ b & b & \cdots & b & b \\ \vdots & \vdots & \cdots & \vdots & \vdots \end{array} \qquad\qquad \boldsymbol{P}' \quad \begin{array}{ccccc} v_1 & v_2 & \cdots & v_{n-1} & v_n \\ \hline b & b & \cdots & b & b \\ a & a & \cdots & a & a \\ \vdots & \vdots & \cdots & \vdots & \vdots \end{array}$$

Pareto $\Rightarrow a\ f(\boldsymbol{P})\ b$ Pareto $\Rightarrow b\ f(\boldsymbol{P}')\ a$

$$
\begin{array}{ccccc}
v_1 & v_2 & \cdots & v_{n-1} & v_n \\
\hline
a & a & \cdots & a & a \\
b & b & \cdots & b & b \\
\vdots & \vdots & \cdots & \vdots & \vdots
\end{array}
$$

$$a \; f(\boldsymbol{P}) \; b$$

$$
\begin{array}{c}
v_1 \\
\hline
b \\
a \\
\vdots
\end{array}
\quad
\begin{array}{ccc}
v_2 & \cdots & v_n \\
\hline
a & \cdots & a \\
b & \cdots & b \\
\vdots & \vdots & \vdots
\end{array}
$$

$$a \; f(\boldsymbol{Q}) \; b$$

$$
\begin{array}{c|ccc}
v_1 & v_2 & \cdots & v_n \\
\hline
b & a & \cdots & a \\
a & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots
\end{array}
\qquad\qquad
\begin{array}{cc|ccc}
v_1 & v_2 & v_3 & \cdots & v_n \\
\hline
b & b & a & \cdots & a \\
a & a & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

$$a \; f(\boldsymbol{P}) \; b \qquad\qquad a \; f(\boldsymbol{Q}) \; b$$

v_1	$\cdots$	v_{i^*-1}	v_{i^*}	v_{i^*+1}	$\cdots$	v_n	
b	$\cdots$	b	a	a	$\cdots$	a	
a	$\cdots$	a	b	b	$\cdots$	b	$a \ f(\boldsymbol{P}) \ b$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

v_1	$\cdots$	v_{i^*-1}	v_{i^*}	v_{i^*+1}	$\cdots$	v_n	
b	$\cdots$	b	b	a	$\cdots$	a	
a	$\cdots$	a	a	b	$\cdots$	b	not $a \ f(\boldsymbol{Q}) \ b$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$	
b	a	a	
a	b	b	$a\ f(\boldsymbol{P})\ b$
$\vdots$	$\vdots$	$\vdots$	

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$	
b	b	a	
a	a	b	not $a\ f(\boldsymbol{Q})\ b$
$\vdots$	$\vdots$	$\vdots$	

$$
\begin{array}{ccc}
v_1 \cdots v_{i^*-1} & v_{i^*} & v_{i^*+1} \cdots v_n \\
\hline
b & a & a \\
a & b & b \\
\vdots & \vdots & \vdots
\end{array}
\qquad a \; f(\boldsymbol{P}) \; b
$$

$$
\begin{array}{ccc}
v_1 \cdots v_{i^*-1} & v_{i^*} & v_{i^*+1} \cdots v_n \\
\hline
b & b & a \\
a & a & b \\
\vdots & \vdots & \vdots
\end{array}
\qquad \text{not } a \; f(\boldsymbol{Q}) \; b
$$

Claim: i^* is decisive for b over c

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b	a	a
a	b	b
$\vdots$	$\vdots$	$\vdots$

$a\ f(\boldsymbol{P})\ b$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b	b	a
a	a	b
$\vdots$	$\vdots$	$\vdots$

not $a\ f(\boldsymbol{Q})\ b$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b	a	a
c	b	b
a	c	c
$\vdots$	$\vdots$	$\vdots$

IIA and $\boldsymbol{P}_{|\{a,b\}} = \boldsymbol{Q}'_{|\{a,b\}} \Rightarrow a\ f(\boldsymbol{Q}')\ b$
Pareto $\Rightarrow b\ f(\boldsymbol{Q}')\ c$
Transitivity $\Rightarrow a\ f(\boldsymbol{Q}')\ c$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b	b	a
a	a	b
$\vdots$	$\vdots$	$\vdots$

not $a\ f(\boldsymbol{Q})\ b$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b	a	a
c	b	b
a	c	c
$\vdots$	$\vdots$	$\vdots$

IIA and $\boldsymbol{P}_{|\{a,b\}} = \boldsymbol{Q}'_{|\{a,b\}} \Rightarrow a\ f(\boldsymbol{Q}')\ b$
Pareto $\Rightarrow b\ f(\boldsymbol{Q}')\ c$
Transitivity $\Rightarrow a\ f(\boldsymbol{Q}')\ c$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b/c	a	a
a	b	b/c
$\vdots$	c	$\vdots$
$\vdots$	$\vdots$	$\vdots$

IIA and $\boldsymbol{Q}'_{|\{a,c\}} = \boldsymbol{Q}''_{|\{a,c\}} \Rightarrow a\ f(\boldsymbol{Q}'')\ c$
IIA and $\boldsymbol{Q}_{|\{a,b\}} = \boldsymbol{Q}''_{|\{a,b\}}$
$\Rightarrow$ not $a\ f(\boldsymbol{Q}'')\ b$
Negative Transitivity $\Rightarrow b\ f(\boldsymbol{Q}'')\ c$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
b/c	a	a
a	b	b/c
$\vdots$	c	$\vdots$
$\vdots$	$\vdots$	$\vdots$

IIA and $\boldsymbol{Q}'_{|\{a,c\}} = \boldsymbol{Q}''_{|\{a,c\}} \Rightarrow a\ f(\boldsymbol{Q}'')\ c$
IIA and $\boldsymbol{Q}_{|\{a,b\}} = \boldsymbol{Q}''_{|\{a,b\}}$
$\Rightarrow$ not $a\ f(\boldsymbol{Q}'')\ b$
Negative Transitivity $\Rightarrow b\ f(\boldsymbol{Q}'')\ c$

$v_1 \cdots v_{i^*-1}$	v_{i^*}	$v_{i^*+1} \cdots v_n$
$\vdots$	$\vdots$	$\vdots$
	b	
	c	
$\vdots$	$\vdots$	$\vdots$

IIA and $\boldsymbol{R}_{|\{b,c\}} = \boldsymbol{Q}''_{|\{b,c\}} \Rightarrow b\ f(\boldsymbol{R})\ c$

Escaping impossibility

Key assumptions in Arrow's Theorem:

- The number of voters is finite

  P. Fishburn (1970). *Arrow's impossibility theorem: concise proof and infinitely many voters*. Journal of Economic Theory, 2, pp. 103 - 106.

- Universal domain

  W. Gaertner (2001). *Domain Conditions in Social Choice Theory*. Cambridge University Press.

  E. Elkind, M. Lackner, and D. Peters (2022). *Preference Restrictions in Computational Social Choice: A Survey*. `https://arxiv.org/abs/2205.09092`.

- There are at least 3 alternatives
- IIA

...one should present a means by which effective and legitimate governance can be reconciled with the fact that what one might term “populist aggregation” will necessarily, in some cases, produce an intransitive or cyclic collective ranking. In other words, there is a need for a theory of how one can take the next step of refining a potentially cyclic collective ranking in order to select a policy to be implemented. (Patty and Penn, Legitimacy and Social Choice)

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Examples of CCRs

- Copeland
- Uncovered Set
- Beat Path
- Split Cycle
- Borda defeat
- ...

Social choice correspondence

A **voting method** is a function F on the domain of all profiles such that for any profile $\mathbf{P}$, $\varnothing \neq F(\mathbf{P}) \subseteq X(\mathbf{P})$ (also called a **variable social choice correspondence** VSCC).

- A (V, X)-SCC is a social choice correspondence defined on (V, X)-profiles.
- A voting method F is **resolute** if for all $\mathbf{P}$, $|F(\mathbf{P})| = 1$. Resolute SCCs are called **social choice functions**.

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There are many examples of voting methods.

See `https://pref_voting.readthedocs.io` for a Python package that provides computational tools to study different voting methods.