New Perspectives on Social Choice

Eric Pacuit, University of Maryland

Lecture 3: Independence of Irrelevant Alternatives

PHIL 808K

Collective choice rules

Definition

A (V, X)-**collective choice rule** (or (V, X)-CCR) is a function f such that for any (V, X)-profile $\boldsymbol{P} \in \mathrm{dom}(f)$, $f(\boldsymbol{P})$ is a binary relation on X.

Collective choice rules

Definition

A (V, X)-**collective choice rule** (or (V, X)-CCR) is a function f such that for any (V, X)-profile $\boldsymbol{P} \in \mathrm{dom}(f)$, $f(\boldsymbol{P})$ is a binary relation on X.

We assume that $f(\boldsymbol{P})$ is at least asymmetric, reflecting our interpretation of $(x, y) \in f(\boldsymbol{P})$ as meaning that x is *strictly socially preferred* to y, or x *defeats* y.

Independence of Irrelevant Alternatives

(IIA): for all (V, X)-profiles $\boldsymbol{P}, \boldsymbol{P}' \in dom(f)$ and $x, y \in X$, if $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then $xf(\boldsymbol{P})y$ if and only if $xf(\boldsymbol{P}')y$.

Independence of Irrelevant Alternatives

(IIA): for all (V, X)-profiles $\boldsymbol{P}, \boldsymbol{P}' \in dom(f)$ and $x, y \in X$, if $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then $xf(\boldsymbol{P})y$ if and only if $xf(\boldsymbol{P}')y$.

(IIA): For all (V, X)-profiles $\boldsymbol{P}, \boldsymbol{P}' \in \mathrm{dom}(f)$ and $x, y \in X$, if $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then $f(\boldsymbol{P})_{|\{x,y\}} = f(\boldsymbol{P}')_{|\{x,y\}}$.

Independence of Irrelevant Alternatives

(IIA): for all (V, X)-profiles $\mathbf{P}, \mathbf{P}' \in dom(f)$ and $x, y \in X$, if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P}')y$.

(IIA): For all (V, X)-profiles $\mathbf{P}, \mathbf{P}' \in \mathrm{dom}(f)$ and $x, y \in X$, if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then $f(\mathbf{P})_{|\{x,y\}} = f(\mathbf{P}')_{|\{x,y\}}$.

(IIA): For all (V, X)-profiles $\mathbf{P} \in \mathrm{dom}(f)$ and all $x, y \in X$, if $\mathbf{P}'$ is a (V, X)-profile such that $\mathbf{P}' \in dom(f)$ and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then

- If x defeats y according to f in $\mathbf{P}$, then x defeats y according to f in $\mathbf{P}'$
- If x does not defeat y according to f in $\mathbf{P}$, then x does not defeats x according to f in $\mathbf{P}'$

Example 1

P:

45	55	$f_{borda}(P')$
a	b	a
c	a	b
b	c	c

P':

45	55	$f_{borda}(P')$
a	b	b
b	a	a
c	c	c

Example 1

P:

45	55	$f_{borda}(P')$
a	b	a
c	a	b
b	c	c

P':

45	55	$f_{borda}(P')$
a	b	b
b	a	a
c	c	c

$\boldsymbol{P}_{|\{a,b\}} = \boldsymbol{P}'_{|\{a,b\}}$, but $a\ f_{borda}(\boldsymbol{P})\ b$ and $b\ f_{borda}(\boldsymbol{P})'\ a$ (and so not-$a\ f_{borda}(\boldsymbol{P})'\ b$)

Example 2

$$
\boldsymbol{P}: \quad
\begin{array}{cc}
1 & 1 \\
\hline
a & c \\
b & b \\
c & a \\
d & d
\end{array}
\qquad
\begin{array}{c}
f_{borda}(\boldsymbol{P}') \\
\hline
a\ b\ c \\
d \\
\\
\\
\end{array}
$$

$$
\boldsymbol{P}': \quad
\begin{array}{cc}
1 & 1 \\
\hline
a & c \\
b & b \\
d & a \\
c & d
\end{array}
\qquad
\begin{array}{c}
f_{borda}(\boldsymbol{P}') \\
\hline
a\ b \\
c \\
d \\
\\
\end{array}
$$

Example 2

P:

1	1
a	c
b	b
c	a
d	d

$f_{borda}(\boldsymbol{P})$
a b c
d

P':

1	1
a	c
b	b
d	a
c	d

$f_{borda}(\boldsymbol{P}')$
a b
c
d

$$\boldsymbol{P}_{|\{b,c\}} = \boldsymbol{P}'_{|\{b,c\}}, \text{ but not-}b\ f_{borda}(\boldsymbol{P})\ c, \text{ not-}c\ f(\boldsymbol{P})\ b, \text{ and } b\ f_{borda}(\boldsymbol{P})'\ c$$

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\boldsymbol{P}|_S$ denote the restriction of the profile to candidates in S.

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\boldsymbol{P}|_S$ denote the restriction of the profile to candidates in S.

Binary Independence: For all profiles $\boldsymbol{P}, \boldsymbol{P}'$ and candidates $A, B \in X$:

$$\text{If } \boldsymbol{P}_{|\{a,b\}} = \boldsymbol{P}'_{|\{a,b\}}, \text{ then } f(\boldsymbol{P})_{|\{a,b\}} = f(\boldsymbol{P}')_{|\{a,b\}}$$

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\boldsymbol{P}|_S$ denote the restriction of the profile to candidates in S.

Binary Independence: For all profiles $\boldsymbol{P}, \boldsymbol{P}'$ and candidates $A, B \in X$:

$$\text{If } \boldsymbol{P}_{|\{a,b\}} = \boldsymbol{P}'_{|\{a,b\}}, \text{ then } f(\boldsymbol{P})_{|\{a,b\}} = f(\boldsymbol{P}')_{|\{a,b\}}$$

m-**Ary Independence**: For all profiles $\boldsymbol{P}, \boldsymbol{P}'$ and for all $S \subseteq X$ with $_{|S|} = m$:

$$\text{If } \boldsymbol{P}_{|S} = \boldsymbol{P}'_{|S}, \text{ then } f(\boldsymbol{P})_{|S} = f(\boldsymbol{P}')_{|S}$$

Weakening IIA

Theorem. (Blau) Suppose that $m = 2, \ldots, |X| - 1$. If a social welfare function F satisfies m-ary independence, then it also satisfies binary independence.

J. Blau. *Arrow's theorem with weak independence*. Economica, 38, pgs. 413 - 420, 1971.

S. Cato. *Independence of Irrelevant Alternatives Revisited*. Theory and Decision, 2013.

Let $\mathcal{S} \subseteq \wp(X)$. F is $\mathcal{S}$-**independent** if for all profiles $\boldsymbol{P}, \boldsymbol{P}'$, and all $S \in \mathcal{S}$,

$$\text{if } \boldsymbol{P}_{|S} = \boldsymbol{P}'_{|S}, \text{ then } f(\boldsymbol{P})_{|S} = f(\boldsymbol{P}')_{|S}$$

$\mathcal{S} \subseteq \wp(X)$ is **connected** provided for all $x, y \in X$ there is a finite set $S^1, \ldots, S^k \in \mathcal{S}$ such that

$$\{x, y\} = \bigcap_{j \in \{1, \ldots, k\}} S^j$$

Let $\mathcal{S} \subseteq \wp(X)$. F is $\mathcal{S}$-**independent** if for all profiles $\boldsymbol{P}, \boldsymbol{P}'$, and all $S \in \mathcal{S}$,

$$\text{if } \boldsymbol{P}_{|S} = \boldsymbol{P}'_{|S}, \text{ then } f(\boldsymbol{P})_{|S} = f(\boldsymbol{P}')_{|S}$$

$\mathcal{S} \subseteq \wp(X)$ is **connected** provided for all $x, y \in X$ there is a finite set $S^1, \ldots, S^k \in \mathcal{S}$ such that

$$\{x, y\} = \bigcap_{j \in \{1,\ldots,k\}} S^j$$

Theorem (Sato). (i) Suppose that $\mathcal{S} \subseteq \wp(X)$ is connected. If a collective choice rule f satisfies $\mathcal{S}$-independence, then it also satisfies binary independence.

(ii) Suppose that $\mathcal{S} \subseteq \wp(X)$ is not connected. Then, there exists a social welfare function f that satisfies $\mathcal{S}$-independence and weak Pareto but does not satisfy binary independence.

Suppose that an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the candidates dies. Surely the social choice should be made by taking each individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining names in going through the procedure of determining the winner. That is, the choice to be made among the set S of surviving candidates should be independent of the preferences of individuals for candidates not in S. To assume otherwise would be to make the result of the election dependent on the obviously accidental circumstance of whether a candidate died before or after the date of polling. (Arrow, 1963, p. 26)

Is Arrow confused?

B. Hansson. *The independence condition in the theory of social choice*. Theory and Decision, 4:25 - 49, 1973.

P. Ray. *Independence of irrelevant alternatives*. Econometrica, 41(5):987 - 991, 1973.

K. Suzumura. *Rational choice, collective decisions, and social welfare*. Cambridge University Press, Cambridge, 1983.

- Variable collective choice rules
- Functional collective choice rules

Variable-Election Profiles

Fix infinite sets $\mathcal{V}$ and $\mathcal{X}$ of *voters* and *candidates*, respectively.

For $X \subseteq \mathcal{X}$, let $\mathcal{L}(X)$ be the set of all strict linear orders on X.

A *profile* is a function $\boldsymbol{P} : V(\boldsymbol{P}) \to \mathcal{L}(X(\boldsymbol{P}))$ for some nonempty finite $V(\boldsymbol{P}) \subseteq \mathcal{V}$ and nonempty finite $X(\boldsymbol{P}) \subseteq \mathcal{X}$.

We call $V(\boldsymbol{P})$ and $X(\boldsymbol{P})$ the sets of *voters in* $\boldsymbol{P}$ and *candidates in* $\boldsymbol{P}$, respectively.

We call $\boldsymbol{P}(i)$ voter i's *ranking*, and we write '$x\boldsymbol{P}_i y$' for $(x, y) \in \boldsymbol{P}(i)$. As usual, we take $x\boldsymbol{P}_i y$ to mean that voter i strictly prefers candidate x to candidate y.

A **variable-election collective choice rule** (VCCR) is a function f on the domain of all profiles such that for any profile $\boldsymbol{P}$, $f(\boldsymbol{P})$ is an asymmetric binary relation on $X(\boldsymbol{P})$, which we call *the defeat relation for* $\boldsymbol{P}$ *under* f.

For $x, y \in X(\boldsymbol{P})$, we say that x *defeats* y *in* $\boldsymbol{P}$ *according to* f when $(x, y) \in f(\boldsymbol{P})$.

Fixed vs. Variable-Candidate Axioms

f satisfies **fixed-candidate IIA** (FIIA) if for any profiles $\mathbf{P}$ and $\mathbf{P}'$ with $X(\mathbf{P}) = X(\mathbf{P}')$,

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in $\mathbf{P}$ according to f if and only if x defeats y in $\mathbf{P}'$ according to f;

Fixed vs. Variable-Candidate Axioms

f satisfies **fixed-candidate IIA** (FIIA) if for any profiles $\mathbf{P}$ and $\mathbf{P}'$ with $X(\mathbf{P}) = X(\mathbf{P}')$,

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in $\mathbf{P}$ according to f if and only if x defeats y in $\mathbf{P}'$ according to f;

f satisfies **variable-candidate IIA** (VIIA) if for any profiles $\mathbf{P}$ and $\mathbf{P}'$, for all $x, y \in X(\mathbf{P}) \cap X(\mathbf{P}')$,

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in $\mathbf{P}$ according to f if and only if x defeats y in $\mathbf{P}'$ according to f.

Fixed vs. Variable-Candidate Axioms

f satisfies **fixed-candidate IIA** (FIIA) if for any profiles $\mathbf{P}$ and $\mathbf{P}'$ with $X(\mathbf{P}) = X(\mathbf{P}')$,

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in $\mathbf{P}$ according to f if and only if x defeats y in $\mathbf{P}'$ according to f;

f satisfies **variable-candidate IIA** (VIIA) if for any profiles $\mathbf{P}$ and $\mathbf{P}'$, for all $x, y \in X(\mathbf{P}) \cap X(\mathbf{P}')$,

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in $\mathbf{P}$ according to f if and only if x defeats y in $\mathbf{P}'$ according to f.

Example 1 - Borda Violates VIIA

$\boldsymbol{P}$:

45	55	$f_{borda}(\boldsymbol{P}')$
a	b	b
b	a	a

$\boldsymbol{P}'$:

45	55	$f_{borda}(\boldsymbol{P}')$
a	b	a
c	a	b
b	c	c

Example 1 - Borda Violates VIIA

P:

45	55	$f_{borda}(P')$
a	b	b
b	a	a

P':

45	55	$f_{borda}(P')$
a	b	a
c	a	b
b	c	c

$$P_{|\{a,b\}} = P'_{|\{a,b\}},\ a, b \in X(P) \cap X(P'),\ \text{but } b\ f_{borda}(P)\ a \text{ and } a\ f_{borda}(P')\ b$$

Arrow's Statement of IIA

SEC. 3] THE INDEPENDENCE OF IRRELEVANT ALTERNATIVES 27

vidual values are given by the first set of orderings as they are when given by the second.

CONDITION 3: *Let* $R_1, \cdots, R_n$ *and* $R_1', \cdots, R_n'$ *be two sets of individual orderings and let* $C(S)$ *and* $C'(S)$ *be the corresponding social choice functions. If, for all individuals* i *and all* x *and* y *in a given environment* S, $x\ R_i\ y$ *if and only if* $x\ R_i'\ y$, *then* $C(S)$ *and* $C'(S)$ *are the same* (independence of irrelevant alternatives).

Choice Consistency

Suppose that C is a choice function on X: for all $\emptyset \neq A \subseteq X$, $\emptyset \neq C(A) \subseteq A$.

Choice Consistency

Suppose that C is a choice function on X: for all $\emptyset \neq A \subseteq X$, $\emptyset \neq C(A) \subseteq A$.

Sen's α condition (independence): if $A' \subseteq A$, then $C(A) \cap A' \subseteq C(A')$

Sen's γ condition (expansion): $C(A) \cap C(A') \subseteq C(A \cup A')$

Choice Consistency

Suppose that C is a choice function on X: for all $\emptyset \neq A \subseteq X$, $\emptyset \neq C(A) \subseteq A$.

Sen's α condition (independence): if $A' \subseteq A$, then $C(A) \cap A' \subseteq C(A')$

Sen's γ condition (expansion): $C(A) \cap C(A') \subseteq C(A \cup A')$

Theorem (Sen 1971)

Let C be a choice function on a nonempty finite set X. TFAE:

1. *C satisfies α and γ*
2. *There exists a binary relation P on X such that for all $A \subseteq X$,*

$$C(A) = \{x \in A \mid \textit{there is no } y \in A \textit{ such that } y \, P \, x\}$$

A. Sen. *Choice Functions and Revealed Preference*. The Review of Economic Studies, 38:3, pp. 307-317, 1971.

Defining a Social Choice Function

A **collective choice rule** (CCR) is a function f on the domain of all profiles such that for any profile $\boldsymbol{P}$, $f(\boldsymbol{P})$ is an asymmetric binary relation on $X(\boldsymbol{P})$.

Defining a Social Choice Function

A **collective choice rule** (CCR) is a function f on the domain of all profiles such that for any profile $\boldsymbol{P}$, $f(\boldsymbol{P})$ is an asymmetric binary relation on $X(\boldsymbol{P})$.

A **voting method** is a function F on the domain of all profiles such that for any profile $\boldsymbol{P}$, $\varnothing \neq F(\boldsymbol{P}) \subseteq X(\boldsymbol{P})$.

Defining a Social Choice Function

A **collective choice rule** (CCR) is a function f on the domain of all profiles such that for any profile $\mathbf{P}$, $f(\mathbf{P})$ is an asymmetric binary relation on $X(\mathbf{P})$.

A **voting method** is a function F on the domain of all profiles such that for any profile $\mathbf{P}$, $\varnothing \neq F(\mathbf{P}) \subseteq X(\mathbf{P})$.

A **functional collective choice rule** (FCCR) is a function F that assigns to each profile $\mathbf{P}$ a choice function $F(\mathbf{P})$ on $X(\mathbf{P})$.

Global vs. Local Choice FCCR

Given an acyclic CCR f, there are two ways to induce an FCCR:

1. the global-choice FCCR $\mathcal{G}_f$: for any profile $\boldsymbol{P}$ and nonempty $Y \subseteq X(\boldsymbol{P})$,

$$\mathcal{G}_f(\boldsymbol{P})(Y) = \{y \in Y \mid \text{there is no } z \in Y \text{ such that } y\ f(\boldsymbol{P})\ x\}.$$

2. the local-choice FCCR $\mathcal{L}_f$: for any profile $\boldsymbol{P}$ and nonempty $Y \subseteq X(\boldsymbol{P})$,

$$\mathcal{L}_f(\boldsymbol{P})(Y) = \{y \in Y \mid \text{there is no } z \in Y \text{ that such that } y\ f(\boldsymbol{P}_{|Y})\ x\}.$$

Global vs. Local Choice FCCR

Given an acyclic CCR f, there are two ways to induce an FCCR:

1. the global-choice FCCR $\mathcal{G}_f$: for any profile $\boldsymbol{P}$ and nonempty $Y \subseteq X(\boldsymbol{P})$,

$$\mathcal{G}_f(\boldsymbol{P})(Y) = \{y \in Y \mid \text{there is no } z \in Y \text{ such that } y\ f(\boldsymbol{P})\ x\}.$$

2. the local-choice FCCR $\mathcal{L}_f$: for any profile $\boldsymbol{P}$ and nonempty $Y \subseteq X(\boldsymbol{P})$,

$$\mathcal{L}_f(\boldsymbol{P})(Y) = \{y \in Y \mid \text{there is no } z \in Y \text{ that such that } y\ f(\boldsymbol{P}_{|Y})\ x\}.$$

Global/Local Borda

Let $Y = \{x, a, y\}$

1	1	2
x	y	y
a	x	x
b	a	c
c	b	b
y	c	a

1	1	2
x	y	y
a	x	x
y	a	a

Global Borda: $\mathcal{G}_{Borda}(\boldsymbol{P})(Y) = \{x\}$

Local Borda: $\mathcal{L}_{Borda}(\boldsymbol{P})(Y) = \{y\}$.

Proposition

Let f be an acyclic VCCR. The following are equivalent:

1. *f satisfies VIIA;*
2. *for any profile $\boldsymbol{P}$ and $Y \subseteq X(\boldsymbol{P})$, $\mathcal{G}_f(\boldsymbol{P}, Y) = \mathcal{L}_f(\boldsymbol{P}, Y)$.*

Definition

Let f be an acyclic VCCR.

1. f satisfies Global-α if $\mathcal{G}_f(\boldsymbol{P}, \cdot)$ satisfies α for all profiles $\boldsymbol{P}$;
2. f satisfies Local-α if $\mathcal{L}_f(\boldsymbol{P}, \cdot)$ satisfies α for all profiles $\boldsymbol{P}$.

Definition

Let f be an acyclic VCCR.

1. f satisfies Global-α if $\mathcal{G}_f(\boldsymbol{P}, \cdot)$ satisfies α for all profiles $\boldsymbol{P}$;
2. f satisfies Local-α if $\mathcal{L}_f(\boldsymbol{P}, \cdot)$ satisfies α for all profiles $\boldsymbol{P}$.

Proposition

If f is an acyclic VCCR, then f satisfies Global-α.

Proposition

1. *If f is an acyclic VCCR satisfying VIIA, then f satisfies Local-α;*
2. *There are acyclic VCCRs satisfying Local-α but not FIIA and hence not VIIA.*

Definition

A VCCR f satisfies *Binary Majoritarianism* if for any profile $\boldsymbol{P}$ with $X(\boldsymbol{P}) = \{x, y\}$, x defeats y in $\boldsymbol{P}$ according to f if and only if x is majority preferred to y in $\boldsymbol{P}$.

Definition

A VCCR f satisfies *Availability* if for any profile $\boldsymbol{P}$, there is some undefeated candidate in $\boldsymbol{P}$ according to f.

Definition

A VCCR f satisfies *Binary Majoritarianism* if for any profile $\boldsymbol{P}$ with $X(\boldsymbol{P}) = \{x, y\}$, x defeats y in $\boldsymbol{P}$ according to f if and only if x is majority preferred to y in $\boldsymbol{P}$.

Definition

A VCCR f satisfies *Availability* if for any profile $\boldsymbol{P}$, there is some undefeated candidate in $\boldsymbol{P}$ according to f.

Proposition

There is no VCCR satisfying Local-α, Availability, and Binary Majoritarianism.

Arrow's Example

> [S]uppose that there are three voters and four candidates, x, y, z, and w. Let the weights for the first, second, third, and fourth choices be 4, 3, 2, and 1, respectively. Suppose that individuals 1 and 2 rank candidates in the order x, y, z, and w, while individual 3 ranks them in the order z, w, x, and y. Under the given electoral system, x is chosen. Then, certainly, if y is deleted from the ranks of the candidates, the system applied to the remaining candidates should yield the same result, especially since, in this case, y is inferior to x according to the tastes of every individual; but, if y is in fact deleted, the indicated electoral system would yield a tie between x and z. (Arrow, 1963, p. 27)

Arrow's Example

Let $\boldsymbol{P}$ be the initial profile described by Arrow with $X(\boldsymbol{P}) = \{x, y, z, w\}$. When Arrow says "if y is deleted from the ranks of the candidates, the system applied to the resulting candidates should yield the same result" which of the following did he mean?

1. since $\mathcal{G}_f(\boldsymbol{P}, X(\boldsymbol{P})) = \{x\}$, we should have $\mathcal{G}_f(\boldsymbol{P}, \{x, z, w\}) = \{x\}$;
2. since $\mathcal{G}_f(\boldsymbol{P}, X(\boldsymbol{P})) = \{x\}$, we should have $\mathcal{G}_f(\boldsymbol{P}_{|\{x,z,w\}}, \{x, z, w\}) = \{x\}$;
3. since $\mathcal{L}_f(\boldsymbol{P}, X(\boldsymbol{P})) = \{x\}$, we should have $\mathcal{L}_f(\boldsymbol{P}, \{x, z, w\}) = \{x\}$;
4. since $\mathcal{L}_f(\boldsymbol{P}, X(\boldsymbol{P})) = \{x\}$, we should have $\mathcal{L}_f(\boldsymbol{P}_{|\{x,z,w\}}, \{x, z, w\}) = \{x\}$.

Arrow could not have meant 2, 3, or 4.

Arrow could not have meant 2, 3, or 4.

Thus, only option 1 officially makes sense in Arrow's framework. Yet if f is Borda count, then $\mathcal{G}_f$ is global Borda count, which still chooses x as the unique winner after y is removed from the input to $\mathcal{G}_f(\boldsymbol{P}, \cdot)$, contradicting Arrow's conclusion.

Arrow could not have meant 2, 3, or 4.

Thus, only option 1 officially makes sense in Arrow's framework. Yet if f is Borda count, then $\mathcal{G}_f$ is global Borda count, which still chooses x as the unique winner after y is removed from the input to $\mathcal{G}_f(\boldsymbol{P}, \cdot)$, contradicting Arrow's conclusion.

Arrow's passage above certainly shows that the Borda VCCR f violates VIIA, because x defeats z in $\boldsymbol{P}$ according to f but not in $\boldsymbol{P}_{|\{x,z,w\}}$. Thus, one way of understanding Arrow's intention in using the example to motivate IIA is that he had in mind VIIA.

M. Fleurbaey (2006). *Social Choice and Just Institutions: New Perspectives*. Economics and Philosophy.

I do not want to be understood as saying that IIA is an axiom based on unsound principles. It would certainly be a good thing if the social comparison of two options could depend only on individual preferences on these two options and on nothing else. That would make social choice a very simple matter. Very little information would be needed in the aggregation process. But simplicity and informational parsimony are not all that counts. Ethical relevance is also important. *And IIA does wipe out ethically relevant information for the social comparison of two options.* (emphasis added) (Fleurbaey, 2006, p. 19)

Example

Suppose that Ann and Bob, who, in the status quo, have the following bundles at their disposal: Ann has 10 apples and 2 oranges, while Bob has 3 apples and 11 oranges.

Would it be a good thing to transfer 1 apple and 1 orange from Bob to Ann?

	Ann	Bob
status quo	$(10, 2)$	$(3, 11)$
after transfer	$(11, 3)$	$(2, 10)$

	Ann	Bob
status quo	$(10, 2)$	$(3, 11)$
after transfer	$(11, 3)$	$(2, 10)$

- Ann prefers the transfer and Bob prefers the status quo. "... on the basis of IIA, [this is] enough information for deciding whether social preferences should approve the transfer, prefer the status quo, or be indifferent."

	Ann	Bob
status quo	$(10, 2)$	$(3, 11)$
after transfer	$(11, 3)$	$(2, 10)$

- There is a perfect symmetry between goods and between people, which makes it impossible to find any reason to prefer one option over the other. But, there may be other relevant information that should influence social preferences:

  > Suppose that Ann and Bob are indifferent between the bundle $(10, 2)$ and the bundle $(3, 11)$. Then, in this profile, the status quo provides Ann and Bob with bundles which both find equally valuable, whereas the transfer would make Bob unambiguously worse-off than Ann, since both would agree that his bundle would be less valuable, and he would *envy* (i.e., he would rather have Ann's bundle).

	Ann	Bob
status quo	$(10, 2)$	$(3, 11)$
after transfer	$(11, 3)$	$(2, 10)$

- Reasonable social preferences may certainly prefer the status quo on these grounds: An efficient envy-free allocation, when individuals have identical preferences, is better than an inefficient allocation with envy.

In other words, it would be quite sensible for social preferences to rely on such information as "Bob prefers Ann's bundle", or Pareto-efficiency of the allocation. But this is, unjustifiably, forbidden by IIA. Notice that the kind of information that this example shows to be relevant belongs only to non-comparable ordinal preferences.

A Non-Economic Example

Consider two options, x and y and two individuals, Alice and Brian. Suppose we know that Alice prefers x and Brian prefers y. According to IIA, this is enough information to determine the social preferences over x and y. What can social preferences be on such a poor informational basis? They should probably declare indifference....

A Non-Economic Example

Consider two options, x and y and two individuals, Alice and Brian. Suppose we know that Alice prefers x and Brian prefers y. According to IIA, this is enough information to determine the social preferences over x and y. What can social preferences be on such a poor informational basis? They should probably declare indifference....

But suppose we now add the information that, with profile $\boldsymbol{P}^*$, in x and y Alice is at her top and second-best options, whereas Brian is at his worst and best options, respectively. Then it might become sensible to prefer y.

A Non-Economic Example

Consider two options, x and y and two individuals, Alice and Brian. Suppose we know that Alice prefers x and Brian prefers y. According to IIA, this is enough information to determine the social preferences over x and y. What can social preferences be on such a poor informational basis? They should probably declare indifference....

But suppose we now add the information that, with profile $\boldsymbol{P}^*$, in x and y Alice is at her top and second-best options, whereas Brian is at his worst and best options, respectively. Then it might become sensible to prefer y.

Again, this additional information about the ranks of options in individuals' preferences may be deemed relevant by reasonable social preferences, and it seems questionable to exclude it, as IIA does.

Weak IIA: Social preferences on a pair of options should only depend on the population's preferences on these two options and on what options are indifferent to each of these two options for each individual.

Weak IIA: Social preferences on a pair of options should only depend on the population's preferences on these two options and on what options are indifferent to each of these two options for each individual.

Knowing the individuals' indifference curves at two allocations makes it possible to answer the following questions:

- Are the individuals indifferent between their respective bundles? In the other allocation, does one individual *envy* the other?
- Is an allocation inefficient in the sense that it would be possible to redistribute its resources in order to make everyone better-off in the population?

Sen: Informational Basis of Social Choice

A. Sen (1999). *The possibility of social choice*. American Economic Review 89: 349 - 78.

...social choice is impossible in the absence of interpersonally comparable indices of well-being.

Utility Functions

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

Utility Functions

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

A preference ordering is **represented** by a utility function iff x is (weakly) preferred to y provided $u(x) \geq u(y)$

L. Narens and B. Skyrms . *The Pursuit of Happiness Philosophical and Psychological Foundations of Utility*. Oxford University Press, 2020.

Let X and V be nonempty sets with $|X| \geq 3$ and V finite.

Let $\mathcal{U}(X)$ be the set of all functions $u : X \to \mathbb{R}$

A **profile** is a function $\boldsymbol{U} : V \to \mathcal{U}(X)$, write $\boldsymbol{U}_i$ for voter i's utility function on X in profile $\boldsymbol{U}$.

A **Social Welfare Functional (SWFL)** is a function f mapping profiles of utilities to asymmetric relations on X. So for each profile $\boldsymbol{U}$, $f(\boldsymbol{U})$ is the social preference order on X.

Arrow Axioms

Universal Domain: the domain of f is the set of all profiles

Weak Pareto: For all $\boldsymbol{U}$ in the domain of f, for all $x, y \in X$, if $\boldsymbol{U}_i(x) > \boldsymbol{U}_i(y)$ for all $i \in V$, then x is ranked strictly above y according to $f(\boldsymbol{U})$.

Independence of Irrelevant Alternatives: For all $\boldsymbol{U}$ and $\boldsymbol{U}'$ in the domain of f, for all $x, y \in X$, if $\boldsymbol{U}_i(x) = \boldsymbol{U}'_i(x)$ and $\boldsymbol{U}_i(y) = \boldsymbol{U}'_i(y)$ for all $i \in V$, then $xf(\boldsymbol{U})y$ if and only if $xf(\boldsymbol{U}')y$

Transitivity/Completeness: For all $\boldsymbol{U}$ in the domain of f, $f(\boldsymbol{U})$ is transitive/complete.

Sum Utilitarian: Define f_S as follows: For all $x, y \in X$, $x f_S(\boldsymbol{U}) y$ if and only if $\sum_i \boldsymbol{U}_i(x) \geq \sum_i \boldsymbol{U}_i(y)$

Lexicographic Maximin: Define f_M as follows: For all $x, y \in X$, $x f_M(\boldsymbol{U}) y$ if and only if $\min_i\{\boldsymbol{U}_i(x)\} \geq \min_i\{\boldsymbol{U}_i(y)\}$ (breaking ties lexicographically: e.g., $\langle 9, 3, 1, 2\rangle$ is "less than" $\langle 1, 2, 4, 8\rangle$)

Sum Utilitarian: Define f_S as follows: For all $x, y \in X$, $xf_S(\boldsymbol{U})y$ if and only if $\sum_i \boldsymbol{U}_i(x) \geq \sum_i \boldsymbol{U}_i(y)$

Lexicographic Maximin: Define f_M as follows: For all $x, y \in X$, $xf_M(\boldsymbol{U})y$ if and only if $\min_i\{\boldsymbol{U}_i(x)\} \geq \min_i\{\boldsymbol{U}_i(y)\}$ (breaking ties lexicographically: e.g., $\langle 9, 3, 1, 2\rangle$ is "less than" $\langle 1, 2, 4, 8\rangle$)

Both SWFLs satisfy all of Arrow's axioms, including non-dictatorship!

U	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

$\boldsymbol{U}$	x	y	z
a	3	1	8
b	300	2	1
c	1	4	1

$\boldsymbol{P}$	a	b	c
	z	x	y
	x	y	x z
	y	z	

$\boldsymbol{U}$	x	y	z
a	3	1	8
b	300	2	1
c	1	400	1

$\boldsymbol{P}$	a	b	c
	z	x	y
	x	y	x z
	y	z	

IIA says two things. First, it says that the comparison of two options should depend only on people's preferences, as opposed to the numerical values of utility. Second, it adds that the comparison must rely only on pairwise preferences about the two contemplated options.

IIA says two things. First, it says that the comparison of two options should depend only on people's preferences, as opposed to the numerical values of utility. Second, it adds that the comparison must rely only on pairwise preferences about the two contemplated options.

Ordinal Non-Comparability: Social preferences should remain the same when the profile of individual utility functions changes without altering individual preferences.

Independence of Irrelevant Utilities: Social preferences on two options should only depend on individual utilities at these two options.

Ordinal Invariance: Given two profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, let $\boldsymbol{U} \sim_{OM} \boldsymbol{U}'$ if for all $i \in V$ and $x, y \in X$, $\boldsymbol{U}_i(x) \geq \boldsymbol{U}_i(y)$ if and only if $\boldsymbol{U}'_i(x) \geq \boldsymbol{U}'_i(y)$.

For all profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, if $\boldsymbol{U} \sim_{OM} \boldsymbol{U}'$, then $f(\boldsymbol{U}) = f(\boldsymbol{U}')$.

Ordinal Invariance: Given two profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, let $\boldsymbol{U} \sim_{OM} \boldsymbol{U}'$ if for all $i \in V$ and $x, y \in X$, $\boldsymbol{U}_i(x) \geq \boldsymbol{U}_i(y)$ if and only if $\boldsymbol{U}'_i(x) \geq \boldsymbol{U}'_i(y)$.

For all profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, if $\boldsymbol{U} \sim_{OM} \boldsymbol{U}'$, then $f(\boldsymbol{U}) = f(\boldsymbol{U}')$.

Cardinal Measurability: Given two profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, let $\boldsymbol{U} \sim_{CM} \boldsymbol{U}'$ if for all $i \in V$, there are $\alpha_i, \beta_i \in \mathbb{R}$ with $\beta_i > 0$ such that for all $x \in X$, $\boldsymbol{U}_i(x) = \alpha_i + \beta_i \boldsymbol{U}'_i(x)$.

For all profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, if $\boldsymbol{U} \sim_{CM} \boldsymbol{U}'$, then $f(\boldsymbol{U}) = f(\boldsymbol{U}')$.

Ordinal Invariance: Given two profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, let $\boldsymbol{U} \sim_{OM} \boldsymbol{U}'$ if for all $i \in V$ and $x, y \in X$, $\boldsymbol{U}_i(x) \geq \boldsymbol{U}_i(y)$ if and only if $\boldsymbol{U}'_i(x) \geq \boldsymbol{U}'_i(y)$.

For all profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, if $\boldsymbol{U} \sim_{OM} \boldsymbol{U}'$, then $f(\boldsymbol{U}) = f(\boldsymbol{U}')$.

Cardinal Measurability: Given two profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, let $\boldsymbol{U} \sim_{CM} \boldsymbol{U}'$ if for all $i \in V$, there are $\alpha_i, \beta_i \in \mathbb{R}$ with $\beta_i > 0$ such that for all $x \in X$, $\boldsymbol{U}_i(x) = \alpha_i + \beta_i \boldsymbol{U}'_i(x)$.

For all profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, if $\boldsymbol{U} \sim_{CM} \boldsymbol{U}'$, then $f(\boldsymbol{U}) = f(\boldsymbol{U}')$.

Cardinal Unit Comparability: Given two profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, let $\boldsymbol{U} \sim_{CUC} \boldsymbol{U}'$ if there is a $\beta \in \mathbb{R}$ with $\beta > 0$ such that for all $i \in V$, there are $\alpha_i \in \mathbb{R}$ such that for all $x \in X$, $\boldsymbol{U}_i(x) = \alpha_i + \beta \boldsymbol{U}'_i(x)$.

For all profiles $\boldsymbol{U}$ and $\boldsymbol{U}'$, if $\boldsymbol{U} \sim_{CUC} \boldsymbol{U}'$, then $f(\boldsymbol{U}) = f(\boldsymbol{U}')$.

To prove his impossibility theorem, Arrow assumed *OM* invariance (and Sen generalized it to *CM* invariance)

Arrow's Epistemological Objection

> The viewpoint will be taken here that interpersonal comparison of utilities has no meaning and, in fact, that there is no meaning relevant to welfare comparisons in the measurability of individual utility...
>
> (Social Choice and Individual Values, p. 9)

Arrow's Epistemological Objection

> Even if...we should admit the measurability of utility for an individual, there is still the question of aggregating the individual utilities.

Arrow's Epistemological Objection

> Even if...we should admit the measurability of utility for an individual, there is still the question of aggregating the individual utilities. At best, it is contended that, for an individual, his utility function is uniquely determined up to a linear transformation; we must still choose one out of the infinite family of indicators to represent the individual, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual.

Arrow's Epistemological Objection

> Even if...we should admit the measurability of utility for an individual, there is still the question of aggregating the individual utilities. At best, it is contended that, for an individual, his utility function is uniquely determined up to a linear transformation; we must still choose one out of the infinite family of indicators to represent the individual, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual. In general, there seems to be no method intrinsic to utility measurement which will make the choices compatible.
>
> (Social Choice and Individual Values, pp. 10-11)

Arrow's Epistemological Objection

It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula....

Arrow's Epistemological Objection

> It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula....If we look away from the mathematical aspects of the matter, it seems to make no sense to add the utility of one individual, a psychic magnitude in his mind, with the utility of another individual. Even Bentham had his doubts on this point.
>
> (Social Choice and Individual Values, pp. 10-11)

Comparability vs. Fairness

Sen and others: Ordinal Non-Comparability must be abandoned, but Independence of Irrelevant Utilities may be kept.

Fleurbaey and others: Retain Ordinal Non-Comparability, which implies that interpersonally comparable utility will be ignored, and non-comparable ordinal preferences will be exclusively considered. But drop Independence of Irrelevant Utilities, and allows the social ranking of two options to depend on features of utilities at other options.

	Ann	Bob
status quo	$(10, 2)$	$(3, 11)$
after transfer	$(11, 3)$	$(2, 10)$

- Suppose that Ann's and Bob's utility levels at the status quo are respectively 4 and 5. And that the transfer of one unit of each good from Bob to Ann would reverse these figures.

- According to Independence of Irrelevant Utilities, this is enough information to make a social decision. Both the utilitarian and the maximin social welfare function, for example, give equal consideration to Ann and Bob and are therefore indifferent between the two options.

	Ann	Bob
status quo	$(10, 2)$	$(3, 11)$
after transfer	$(11, 3)$	$(2, 10)$

- They are therefore unable to take account of the fact that, maybe, the status quo is an efficient allocation in which both agents are indifferent between the two bundles, whereas the transfer would destroy efficiency and would make Bob have a bundle that is then unambiguously worse than Ann's. Such fairness considerations require more information about preferences than allowed by Independence of Irrelevant Utilities.

In the “comparability” approach, comparison is about utility, whatever that is. In the “fairness” approach, the comparison is about bundles, as they are valued by individual preferences, with reference to equity principles. If utility, in the “comparability” approach, is defined in such a way that it merely reflects the value of bundles as assessed through individual preferences, then the two approaches, in their results, come close to each other. If, on the contrary, a more comprehensive notion of well-being is invoked in the comparability approach, then the two approaches lead to social preferences that are substantially different.

H. Greaves and H. Lederman (2017). *Aggregating extended preferences*. Philosophical Studies, 174:1163 - 1190.

A. Khmelnitskaya and J. Weymark (2000). *Social choice with independent subgroup utility scales*. Social Choice and Welfare, 17:4, pp. 739-748.

F. Dietrich. *Welfarism, preferencism, judgmentism*. manuscript, 2006.

Theorem (Patty and Penn, 2014)

Arrow's IIA condition is equivalent to the condition of unilateral flip independence: if two profiles are alike except that one voter flips one pair of adjacent candidates on her ballot, then the defeat relations for the two profiles can differ at most on the flipped candidates.

Theorem (Patty and Penn, 2014)

Arrow's IIA condition is equivalent to the condition of unilateral flip independence: if two profiles are alike except that one voter flips one pair of adjacent candidates on her ballot, then the defeat relations for the two profiles can differ at most on the flipped candidates.

This theorem "demonstrates a fundamental basis of the normative appeal of IIA" (p. 52, Penn and Patty, 2014).

Escaping impossibility

1. Variable-candidate setting

Escaping impossibility

1. Variable-candidate setting

2. Weaken the IIA axiom

Escaping impossibility

1. Variable-candidate setting

2. Weaken the IIA axiom

   Weak IIA: For all profiles $\boldsymbol{P}$ and $\boldsymbol{P}'$, if x defeats y in $\boldsymbol{P}$ according to f and $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then y does *not* defeat x in $\boldsymbol{P}'$ according to f.

3. Weaken properties of the defeat relation

   $x_1, \ldots, x_n$ is a **cycle** in B if $x_1 = x_n$ and for all $i = 1, \ldots, n-1$, $x_i B x_{i+1}$.
   A relation B is **acyclic** if there is no cycle in B.

Baigent's Theorem

Weak IIA: For all profiles $\mathbf{P}$ and $\mathbf{P}'$, if x defeats y in $\mathbf{P}$ according to f and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then y does *not* defeat x in $\mathbf{P}'$ according to f.

Baigent's Theorem

Weak IIA: For all profiles $\boldsymbol{P}$ and $\boldsymbol{P}'$, if x defeats y in $\boldsymbol{P}$ according to f and $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then y does *not* defeat x in $\boldsymbol{P}'$ according to f.

Given a (V, X)-CCR f, a voter $i \in V$ is a *vetoer for* f if for all (V, X)-profiles $\boldsymbol{P}$ and $x, y \in X(\boldsymbol{P})$, if $x\boldsymbol{P}_i y$, then y does not defeat x in $\boldsymbol{P}$ according to f.

Theorem (Baigent, 1987)

Assume V is finite and $|X| \geq 4$. Any (V, X)-SWF satisfying Weak IIA and Pareto has a vetoer.

Blau and Deb Theorem

A coalition $C \subseteq V$ of voters has *veto power for* f if for any (V, X)-profile $\mathbf{P}$ and $x, y \in X$, if $x\mathbf{P}_i y$ for all $i \in C$, then y does not defeat x in $\mathbf{P}$ according to f

Theorem (Blau and Deb, 1977)

Let f be an acyclic (V, X)-CCR satisfying IIA, Neutrality, and Monotonicity.

1. *For any partition of V into at least $|X|$-many coalitions, at least one of the coalitions has veto power.*
2. *If $|X| \geq |V|$, then f has a vetoer.*

Blau and Deb Theorem

A coalition $C \subseteq V$ of voters has *veto power for* f if for any (V, X)-profile $\boldsymbol{P}$ and $x, y \in X$, if $x\boldsymbol{P}_i y$ for all $i \in C$, then y does not defeat x in $\boldsymbol{P}$ according to f

Theorem (Blau and Deb, 1977)

Let f be an acyclic (V, X)-CCR satisfying IIA, Neutrality, and Monotonicity.

1. *For any partition of V into at least $|X|$-many coalitions, at least one of the coalitions has veto power.*
2. *If $|X| \geq |V|$, then f has a vetoer.*

Theorem (Holliday and P, 2021)

If f is an acyclic VCCR satisfying VIIA, Neutrality, and Monotonicity, then for any finite $V \subseteq \mathcal{V}$, f has a V-vetoer.

Modified IIA

Modified IIA: for all profiles $\boldsymbol{P}$ and $\boldsymbol{P}'$, if $\boldsymbol{P}_{\{x,y\}} = \boldsymbol{P}'_{\{x,y\}}$, and for each voter i and candidate z, i ranks z in between x and y in $\boldsymbol{P}$ if and only if i ranks z in between x and y in $\boldsymbol{P}'$, then x defeats y in $\boldsymbol{P}$ if and only if x defeats y in $\boldsymbol{P}'$. (cf. Saari, 1994, 1995, 1998)

n	n	n	n
a	b	a	a
b	c	b	b
c	d	c	d
d	a	d	c

n	n	n	n
a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

E. Maskin (2020). *A Modified Version of Arrow's IIA Condition*. Social Choice and Welfare.

The Fallacy of IIA

Suppose x defeats y in a profile $\boldsymbol{P}$, and a profile $\boldsymbol{P}'$ is exactly like $\boldsymbol{P}$ with respect to how every voter ranks x vs. y. Should it follow that x defeats y in $\boldsymbol{P}'$?

The Fallacy of IIA

Suppose x defeats y in a profile $\boldsymbol{P}$, and a profile $\boldsymbol{P}'$ is exactly like $\boldsymbol{P}$ with respect to how every voter ranks x vs. y. Should it follow that x defeats y in $\boldsymbol{P}'$?

Arrow's Independence of Irrelevant Alternatives (IIA) says 'yes'.

The Fallacy of IIA

Suppose x defeats y in a profile $\mathbf{P}$, and a profile $\mathbf{P}'$ is exactly like $\mathbf{P}$ with respect to how every voter ranks x vs. y. Should it follow that x defeats y in $\mathbf{P}'$?

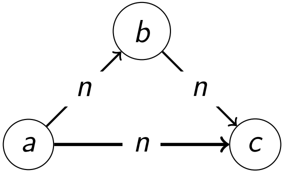
Arrow's Independence of Irrelevant Alternatives (IIA) says 'yes'.

We say 'no': if $\mathbf{P}'$ is sufficiently *incoherent*, we may need to suspend judgment on many defeat relations that could be coherently accepted in $\mathbf{P}$.

W. Holliday and EP (2021). *Axioms for Defeat in Democratic Elections*. Journal of Theoretical Politics.

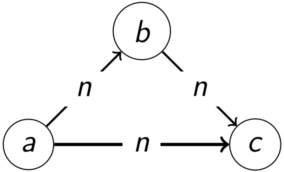
In the context of the following perfectly coherent profile $\boldsymbol{P}$, the margin of n for a over b should be sufficient for a to defeat b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
c	c	$\boldsymbol{b}$



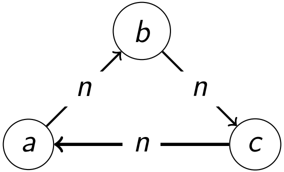
In the context of the following perfectly coherent profile $\boldsymbol{P}$, the margin of n for a over b should be sufficient for a to defeat b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
c	c	$\boldsymbol{b}$



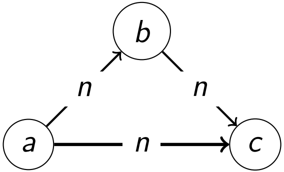
Yet in the following $\boldsymbol{P}'$ with $\boldsymbol{P}'_{|\{a,b\}} = \boldsymbol{P}_{|\{a,b\}}$, no VCCR satisfying Anonymity, Neutrality, and Availability can say that a defeats b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	c	$\boldsymbol{a}$
c	$\boldsymbol{a}$	$\boldsymbol{b}$



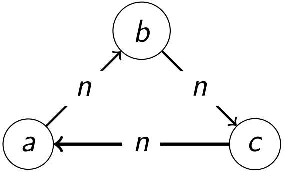
In the context of the following perfectly coherent profile $\boldsymbol{P}$, the margin of n for a over b should be sufficient for a to defeat b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
c	c	$\boldsymbol{b}$



Yet in the following $\boldsymbol{P}'$ with $\boldsymbol{P}'_{|\{a,b\}} = \boldsymbol{P}_{|\{a,b\}}$, no VCCR satisfying Anonymity, Neutrality, and Availability can say that a defeats b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	c	$\boldsymbol{a}$
c	$\boldsymbol{a}$	$\boldsymbol{b}$



This is a counterexample to IIA as a plausible axiom.

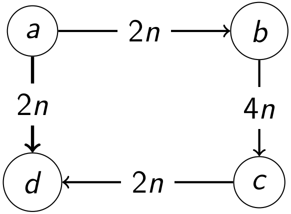
Theorem (Patty and Penn, 2014)

Arrow's IIA condition is equivalent to the condition of unilateral flip independence: if two profiles are alike except that one voter flips one pair of adjacent candidates on her ballot, then the defeat relations for the two profiles can differ at most on the flipped candidates.

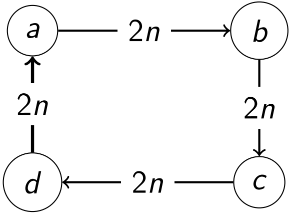
Unilateral flip independence makes the same mistake as IIA in ignoring how context can affect the standard for defeat (let $n = 1$ and consider the middle voter in the previous example).

Modified IIA makes the same mistake as IIA:

n	n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
$\boldsymbol{b}$	c	$\boldsymbol{b}$	$\boldsymbol{b}$
c	d	c	d
d	$\boldsymbol{a}$	d	c



n	n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c	d
$\boldsymbol{b}$	c	d	$\boldsymbol{a}$
c	d	$\boldsymbol{a}$	$\boldsymbol{b}$
d	$\boldsymbol{a}$	$\boldsymbol{b}$	c



Coherent IIA

Coherent IIA: if x defeats y in $\boldsymbol{P}$,

Coherent IIA

Coherent IIA: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is a profile such that

1. $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$

Coherent IIA

Coherent IIA: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is a profile such that

1. $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$ and
2. the margin graph of $\boldsymbol{P}'$ is obtained from that of $\boldsymbol{P}$ by deleting zero or more candidates other than x and y and deleting or reducing the margins on zero or more edges not connecting x and y,

Coherent IIA

Coherent IIA: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is a profile such that

1. $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$ and
2. the margin graph of $\boldsymbol{P}'$ is obtained from that of $\boldsymbol{P}$ by deleting zero or more candidates other than x and y and deleting or reducing the margins on zero or more edges not connecting x and y,

then x still defeats y in $\boldsymbol{P}'$.

Coherent IIA

Coherent IIA: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is a profile such that

1. $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$ and
2. the margin graph of $\boldsymbol{P}'$ is obtained from that of $\boldsymbol{P}$ by deleting zero or more candidates other than x and y and deleting or reducing the margins on zero or more edges not connecting x and y,

then x still defeats y in $\boldsymbol{P}'$.

Key idea: the operations described in 2 cannot increase cyclic incoherence.

Coherent IIA

Coherent IIA: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is a profile such that

1. $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$ and
2. the margin graph of $\boldsymbol{P}'$ is obtained from that of $\boldsymbol{P}$ by deleting zero or more candidates other than x and y and deleting or reducing the margins on zero or more edges not connecting x and y,

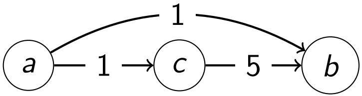
then x still defeats y in $\boldsymbol{P}'$.

Key idea: the operations described in 2 cannot increase cyclic incoherence.

Note: this is a variable-candidate axiom, so it is best compared to what we call VIIA (see our "Axioms for Defeat in Democratic Elections").

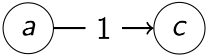
Violations of Coherent IIA

2	3
c	a
b	c
a	b



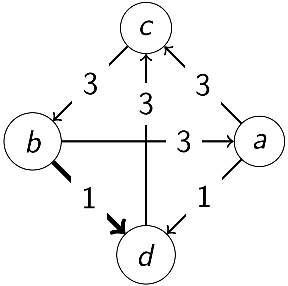
Borda: c defeats a

2	3
c	a
a	c

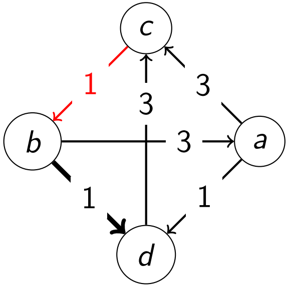


Borda winner: a defeats c

Violations of Coherent IIA



Beat Path: d defeats b



Beat Path: d doesn't defeat b

Coherent IIA and acyclicity

Proposition

Coherent IIA implies Weak IIA.

Coherent IIA and acyclicity

Proposition

Coherent IIA implies Weak IIA.

There is an acyclic VCCR satisfying Coherent IIA: The Split Cycle defeat relation

W. Holliday and EP (2021). *Axioms for Defeat in Democratic Elections*. Forthcoming in Public Choice.

W. Holliday and EP (2022). *Axioms for Defeat in Democratic Elections*. Journal of Theoretical Politics, https://arxiv.org/pdf/2008.08451.pdf.

Y. Ding, W. Holliday and EP (2022). *A Full Characterization of Split Cycle*. manuscript.