New Perspectives on Social Choice

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Lecture 4: Split Cycle

PHIL 808K

Background

Margin

Let P be a profile and $a, b \in X(P)$. Then the **margin of a over b** is:

$$Margin_{\boldsymbol{P}}(a, b) = |\{i \in V(\boldsymbol{P}) \mid a\boldsymbol{P}_i b\}| - |\{i \in V(\boldsymbol{P}) \mid b\boldsymbol{P}_i a\}|.$$

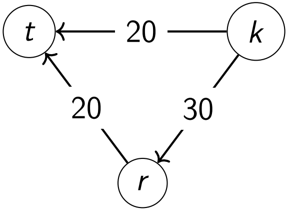
We say that a is **majority preferred** to b in $\boldsymbol{P}$ when $Margin_{\boldsymbol{P}}(a, b) > 0$.

Margin Graph

The **margin graph of P**, $\mathcal{M}(P)$, is the weighted directed graph whose set of nodes is $X(P)$ with an edge from a to b weighted by $Margin(a,b)$ when $Margin(a,b) > 0$. We write

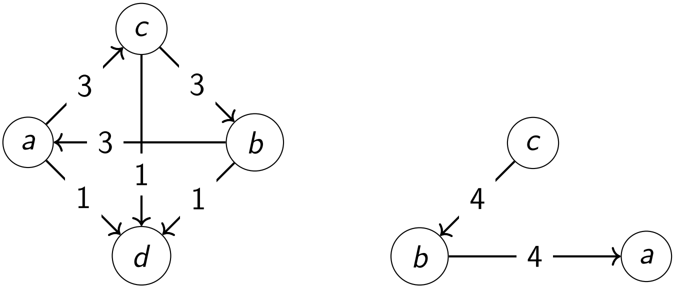
$$a \xrightarrow{\alpha}_{\boldsymbol{P}} b \text{ if } \alpha = Margin_{\boldsymbol{P}}(a,b) > 0.$$

40	35	25
t	r	k
k	k	r
r	t	t



Margin Graph

A **margin graph** is a weighted directed graph $\mathcal{M}$ where all the weights have the same parity.



Theorem (Debord, 1987)

For any margin graph $\mathcal{M}$, there is a profile $\boldsymbol{P}$ such that $\mathcal{M}$ is the margin graph of $\boldsymbol{P}$.

VCCRs, Voting Methods

A **variable collective choice rule** (VCCR) is a function f on the domain of all profiles such that for any profile $\mathbf{P}$, $f(\mathbf{P})$ is an asymmetric binary relation on $X(\mathbf{P})$.

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A **voting method** is a function F on the domain of all profiles such that for any profile $\boldsymbol{P}$, $\varnothing \neq F(\boldsymbol{P}) \subseteq X(\boldsymbol{P})$.

- See `https://pref_voting.readthedocs.io` for a Python package that provides computational tools to study different voting methods.

Every acyclic VCCR f generates a voting method $\overline{f}$, where for all $\boldsymbol{P}$, $\overline{f}(\boldsymbol{P})$ is the set of undefeated candidates in $\boldsymbol{P}$ according to f.

Condorcet criteria

The **Condorcet winner** in a profile $\boldsymbol{P}$ is a candidate $x \in X(\boldsymbol{P})$ that is the maximum of the majority ordering, i.e., for all $y \in X(\boldsymbol{P})$, if $x \neq y$, then $Margin_{\boldsymbol{P}}(x, y) > 0$.

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The **Condorcet loser** in a profile $\mathbf{P}$ is a candidate $x \in X(\mathbf{P})$ that is the minimum of the majority ordering, i.e., for all $y \in X(\mathbf{P})$, if $x \neq y$, then $Margin_{\mathbf{P}}(y, x) < 0$.

A voting method F is susceptible to the **Condorcet loser paradox** (also known as *Borda's paradox*) if there is some $\mathbf{P}$ such that x is a Condorcet loser in $\mathbf{P}$ and $x \in F(\mathbf{P})$.

- Coherent IIA
    - Majority Defeat
- Split Cycle: A VCCR that satisfies Coherent IIA
- Characterizing Split Cycle
    - Stability for Winners
    - Positive Involvement
- Refining Split Cycle: Stable Voting

The Fallacy of IIA

Suppose x defeats y in a profile $\boldsymbol{P}$, and a profile $\boldsymbol{P}'$ is exactly like $\boldsymbol{P}$ with respect to how every voter ranks x vs. y. Should it follow that x defeats y in $\boldsymbol{P}'$?

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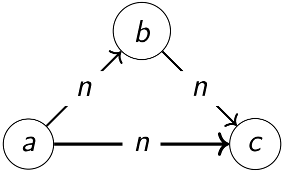
We say 'no': if $\mathbf{P}'$ is sufficiently *incoherent*, we may need to suspend judgment on many defeat relations that could be coherently accepted in $\mathbf{P}$.

W. Holliday and EP (2021). *Axioms for Defeat in Democratic Elections*. Journal of Theoretical Politics.

In the context of the following perfectly coherent profile $\boldsymbol{P}$, the margin of n for a over b should be sufficient for a to defeat b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
c	c	$\boldsymbol{b}$



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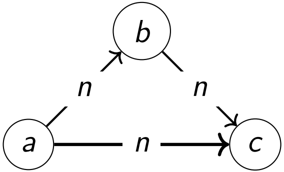

Yet in the following $\boldsymbol{P}'$ with $\boldsymbol{P}'_{|\{a,b\}} = \boldsymbol{P}_{|\{a,b\}}$, no VCCR satisfying Anonymity, Neutrality, and Availability can say that a defeats b:

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$\boldsymbol{b}$	c	$\boldsymbol{a}$
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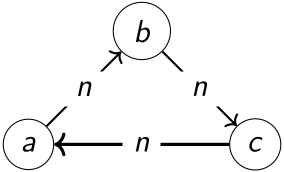

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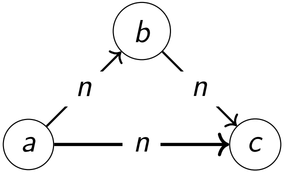

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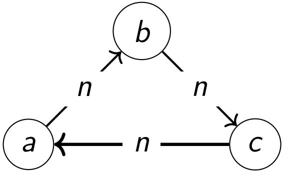

This is a counterexample to IIA as a plausible axiom.

Theorem (Patty and Penn, 2014)

Arrow's IIA condition is equivalent to the condition of unilateral flip independence: if two profiles are alike except that one voter flips one pair of adjacent candidates on her ballot, then the defeat relations for the two profiles can differ at most on the flipped candidates.

Unilateral flip independence makes the same mistake as IIA in ignoring how context can affect the standard for defeat (let $n = 1$ and consider the middle voter in the previous example).

Modified IIA

Modified IIA: for all profiles $\boldsymbol{P}$ and $\boldsymbol{P}'$, if $\boldsymbol{P}_{\{x,y\}} = \boldsymbol{P}'_{\{x,y\}}$, and for each voter i and candidate z, i ranks z in between x and y in $\boldsymbol{P}$ if and only if i ranks z in between x and y in $\boldsymbol{P}'$, then x defeats y in $\boldsymbol{P}$ if and only if x defeats y in $\boldsymbol{P}'$. (cf. Saari, 1994, 1995, 1998)

n	n	n	n
a	b	a	a
b	c	b	b
c	d	c	d
d	a	d	c

n	n	n	n
a	b	c	d
b	c	d	a
c	d	a	b
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E. Maskin (2020). *A Modified Version of Arrow's IIA Condition*. Social Choice and Welfare.

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n	n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
$\boldsymbol{b}$	c	$\boldsymbol{b}$	$\boldsymbol{b}$
c	d	c	d
d	$\boldsymbol{a}$	d	c

$$\begin{tikzcd} a \arrow[r, "2n"] \arrow[d, "2n"'] & b \arrow[d, "4n"] \\ d & c \arrow[l, "2n"] \end{tikzcd}$$

n	n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c	d
$\boldsymbol{b}$	c	d	$\boldsymbol{a}$
c	d	$\boldsymbol{a}$	$\boldsymbol{b}$
d	$\boldsymbol{a}$	$\boldsymbol{b}$	c

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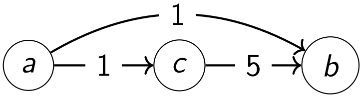
Note: this is a variable-candidate axiom, so it is best compared to what we call VIIA (see our "Axioms for Defeat in Democratic Elections").

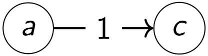
Violations of Coherent IIA

2	3
c	a
b	c
a	b



Borda: c defeats a

2	3
c	a
a	c



Borda winner: a defeats c

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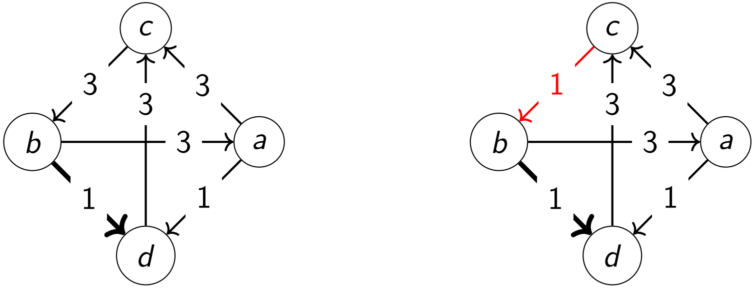

Beat Path: d defeats b

Beat Path: d doesn't defeat b

Coherent IIA

Proposition

Coherent IIA implies Weak IIA.

Weak IIA: For all profiles $\boldsymbol{P}$ and $\boldsymbol{P}'$, if x defeats y in $\boldsymbol{P}$ according to f and $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then y does *not* defeat x in $\boldsymbol{P}'$ according to f.

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Weak IIA: For all profiles $\boldsymbol{P}$ and $\boldsymbol{P}'$, if x defeats y in $\boldsymbol{P}$ according to f and $\boldsymbol{P}_{|\{x,y\}} = \boldsymbol{P}'_{|\{x,y\}}$, then y does *not* defeat x in $\boldsymbol{P}'$ according to f.

There is an acyclic VCCR satisfying Coherent IIA: The Split Cycle defeat relation

W. Holliday and EP (2022). *Axioms for Defeat in Democratic Elections*. Forthcoming in Public Choice.

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Lemma

Anonymity, Neutrality, Monotonicity (for two-candidate profiles), and Coherent IIA together imply:

If x defeats y, then x is majority preferred to y.

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Under Plurality, Nader **spoiled** the election for Gore, handing victory to Bush.

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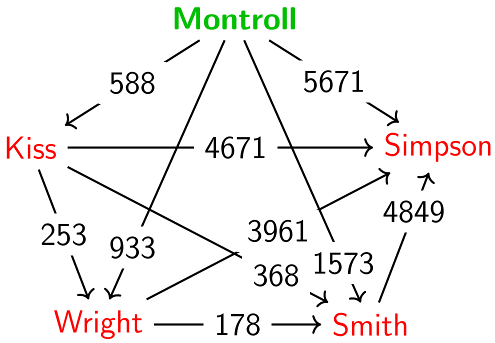

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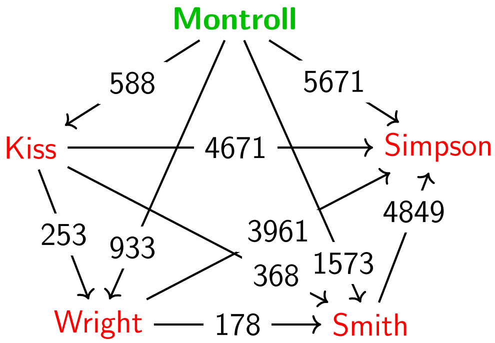

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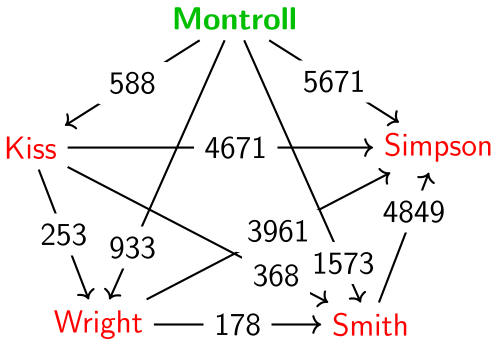

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Montroll was the **Condorcet winner**. IRV was repealed in 2010.

2022 Alaska Special General Election

The IRV winner is Petola.

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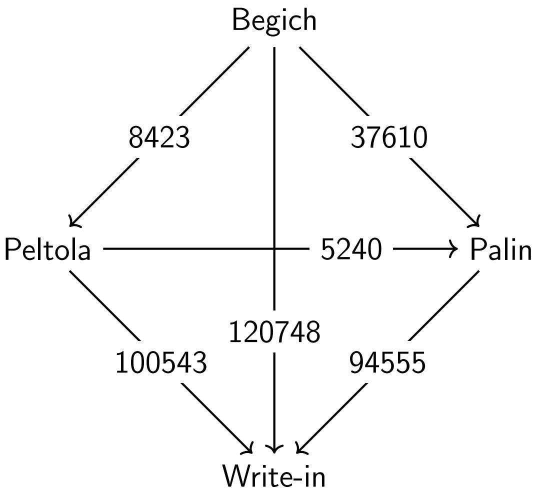

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Maskin and Sen (2017) make the following conjecture:

> [M]ajority rule may reduce polarization. A centrist like Bloomberg [in the 2016 U.S. presidential election] may not be ranked first by a large proportion of voters [and hence cannot win under Plurality], but can still be elected [with the backing of majorities against each other candidate] if viewed as a good compromise. Majority rule also encourages public debate about a larger group of potential candidates [since more candidates can participate without worry of their being spoilers], bringing us closer to John Stuart Mill's ideal of democracy as "government by discussion."

Problem with Majority Defeat: The "Paradox of Voting"

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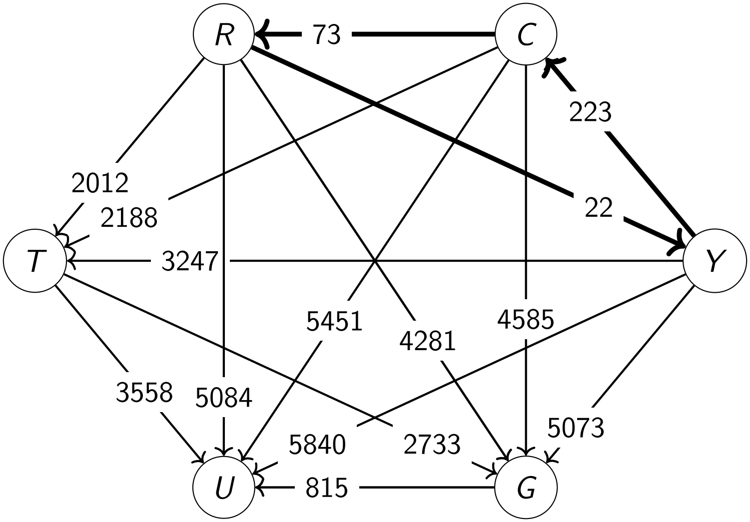
2021 Minneapolis City Council Ward 2 Election



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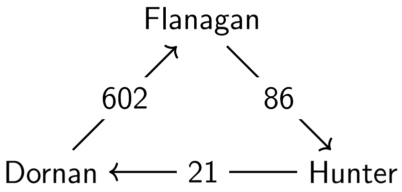

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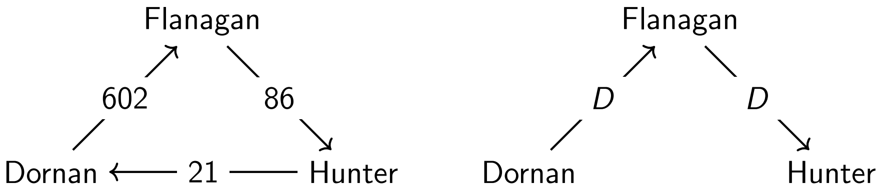
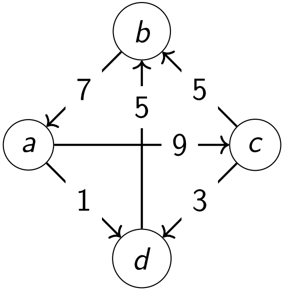


Example

Suppose an election produces the following majority margin graph (i.e., there are 7 more voters who ranked ***b*** above ***a*** than who ranked ***a*** above ***b***, etc.):



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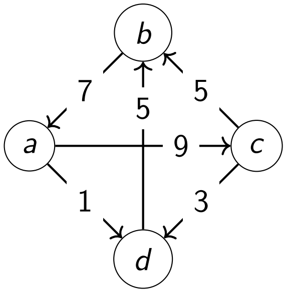
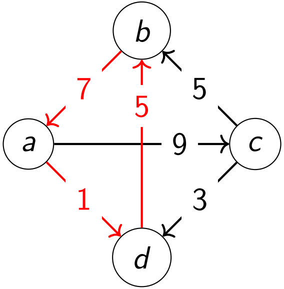
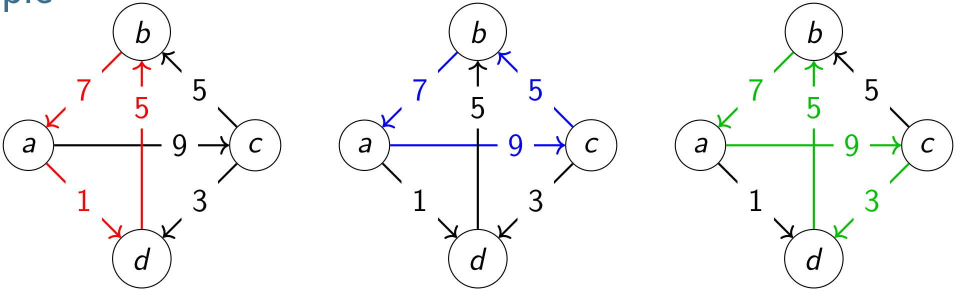


Our first step is to identify the cycles...

Example

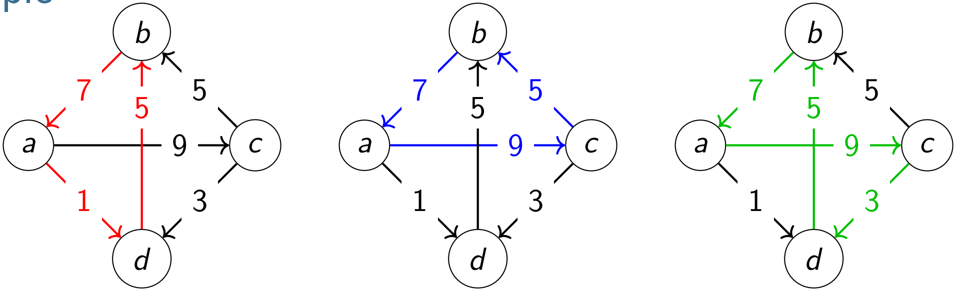


Example



Next find the smallest margin in each cycle.

Example



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These edges cannot be defeats.

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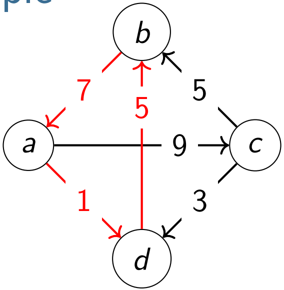
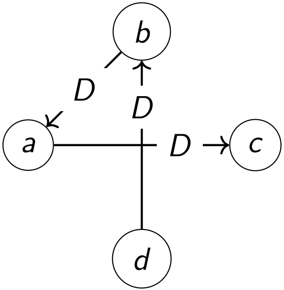


Next find the smallest margin in each cycle.
These edges cannot be defeats.
But all other edges are defeats:



Motivating ideas

Split Cycle can be motivated using three main ideas...

Idea 1

Group *incoherence* raises the threshold for one candidate to defeat another, but not infinitely.

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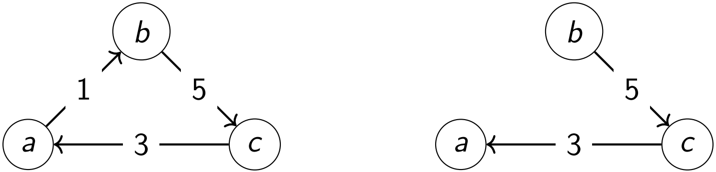

a does not defeat ***b***, but ***b*** defeats ***c*** and ***c*** defeats ***a***.

Idea 2

Incoherence can be localized.

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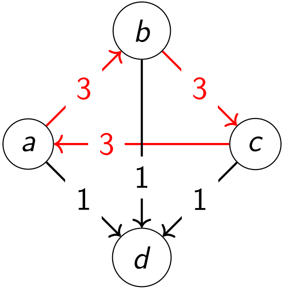
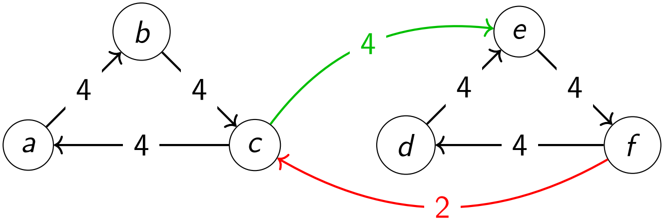


a does not defeat ***b***, but ***a*** defeats ***d***.

Idea 3

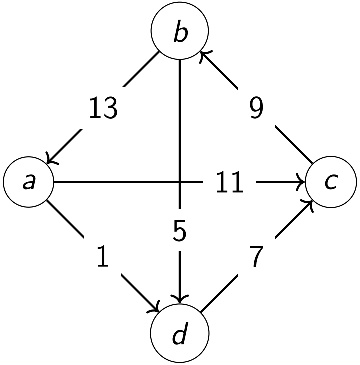
Majority defeat: a candidate should defeat another only if a majority of voters prefer the first candidate to the second. (We also say "Defeat is direct")



$\boldsymbol{c}$ defeats $\boldsymbol{e}$, but $\boldsymbol{c}$ does not defeat $\boldsymbol{f}$.

- ✓ Coherent IIA
  - ✓ Majority Defeat
- ✓ Split Cycle: A VCCR that satisfies Coherent IIA
- Characterizing Split Cycle
  - Stability for Winners
  - Positive Involvement
- Refining Split Cycle: Stable Voting

Condorcet consistent methods



Minimax:	$\{d\}$
Copeland:	$\{a, b\}$
Beat Path:	$\{d\}$
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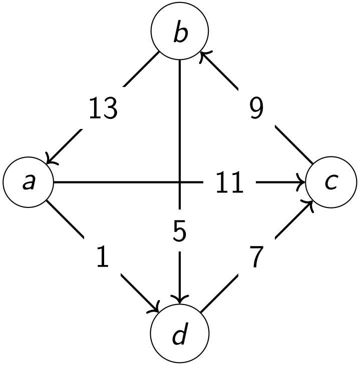

Minimax:	$\{d\}$
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Ranked Pairs:	$\{b\}$
Split Cycle:	$\{b, d\}$

Proposition. Both Ranked Pairs and Beat Path refine Split Cycle (i.e., in all profiles, any Ranked Pairs (resp. Beat Path) winner is also a Split Cycle winner.

Distinguishing Split Cycle from other definitions of defeat

Split Cycle is distinguished from other definitions of defeat by:

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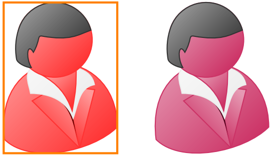
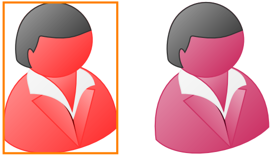
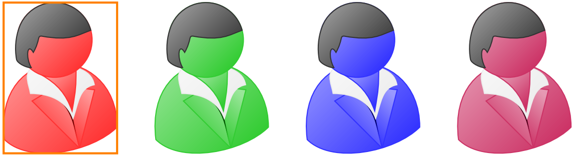
Stability for Winners

If  wins and  wins

Stability for Winners

If  wins

and  wins

then  wins

.

Stability for Winners

Definition

A VCCR satisfies **Stability for Winners** if for any profile $\mathbf{P}$ and $a, b \in X(\mathbf{P})$, if a is undefeated in $\mathbf{P}_{-b}$ and $Margin_{\mathbf{P}}(a, b) > 0$, then a is undefeated in $\mathbf{P}$.

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- there are also violations in profiles with no Condorcet winner.

Proposition

Anonymity, Neutrality, Monotonicity (for two-candidate profiles), and Coherent IIA together imply Stability for Winners*.*

Choice Consistency

Suppose that C is a choice function on X: for all $\emptyset \neq A \subseteq X$, $\emptyset \neq C(A) \subseteq A$.

Sen's α condition: if $A' \subseteq A$, then $C(A) \cap A' \subseteq C(A')$

Sen's γ condition (expansion): $C(A) \cap C(A') \subseteq C(A \cup A')$

Theorem (Sen 1971)

Let C be a choice function on a nonempty finite set X. TFAE:

1. *C satisfies α and γ*
2. *There exists a binary relation P on X such that for all $A \subseteq X$,*

$$C(A) = \{x \in A \mid \textit{there is no } y \in A \textit{ such that } y \, P \, x\}$$

A. Sen. *Choice Functions and Revealed Preference*. The Review of Economic Studies, 38:3, pp. 307-317, 1971.

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Binary Expansion

Expansion: For all $A, A' \subseteq X$, $C(A) \cap C(A') \subseteq C(A \cup A')$.

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Modulo α, Expansion is equivalent to Binary Expansion. Thus, we can replace Expansion by Binary Expansion in Sen's representation theorem.

Expansion in Voting

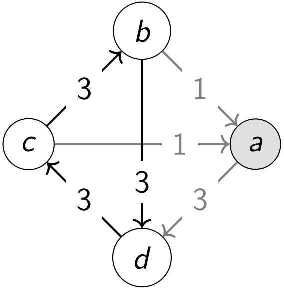
A **voting method** is a function F on the domain of all profiles such that for any profile $\boldsymbol{P}$, $\varnothing \neq F(\boldsymbol{P}) \subseteq X(\boldsymbol{P})$.

A voting method F satisfies Expansion if for all profiles $\boldsymbol{P}$ and Y, Y' with $Y \cup Y' = X(\boldsymbol{P})$,

$$F(\boldsymbol{P}_{|Y}) \cap F(\boldsymbol{P}_{|Y'}) \subseteq F(\boldsymbol{P}).$$

Beat Path and Minimax Violate Binary Expansion

2	1	1	1	3	1	1	1
b	a	d	a	c	b	c	d
a	b	a	d	b	d	d	c
d	d	c	c	a	c	a	a
c	c	b	b	d	a	b	b



Beat Path and Minimax both violate Binary Expansion: $F(\boldsymbol{P}_{-a}) = \{b, c, d\}$, $Margin_{\boldsymbol{P}}(b, a) > 0$ (so $F(\boldsymbol{P}_{|\{a,b\}}) = \{b\}$), and $b \notin F(\boldsymbol{P})$.

Spoilers

Binary Expansion rules out *spoilers*.

37	29	34
d	d	p
p	p	d

IR Winner: d

37	29	34
r	d	p
d	p	d
p	r	r

IR Winner: p

Immunity to Spoilers: For all profiles $\mathbf{P}$ and $a, b \in X(\mathbf{P})$,
if $a \in F(\mathbf{P}_{-b})$, $Margin_{\mathbf{P}}(a, b) > 0$ and $b \notin F(\mathbf{P})$, then $a \in F(\mathbf{P})$

Minimax, Copeland, and GOCHA all satisfy Immunity to Spoilers, but not Binary Expansion

Spoilers, Stealers

Definition

Let F be a voting method, $\boldsymbol{P} \in \mathrm{dom}(F)$, and $a, b \in X(\boldsymbol{P})$. Then we say that:

1. *b spoils the election for a in $\boldsymbol{P}$*
   if $a \in F(\boldsymbol{P}_{-b})$, $Margin_{\boldsymbol{P}}(a, b) > 0$, $a \notin F(\boldsymbol{P})$, and $b \notin F(\boldsymbol{P})$;
2. *b steals the election from a in $\boldsymbol{P}$*
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Spoilers, Stealers

Definition

Let F be a voting method.

1. F satisfies *immunity to spoilers* if for $\boldsymbol{P} \in \mathrm{dom}(F)$ and $a, b \in X(\boldsymbol{P})$, b does not spoil the election for a.
2. F satisfies *immunity to stealers* if for $\boldsymbol{P} \in \mathrm{dom}(F)$ and $a, b \in X(\boldsymbol{P})$, b does not steal the election from a.
3. F satisfies *stability for winners* if for $\boldsymbol{P} \in \mathrm{dom}(F)$ and $a, b \in X(\boldsymbol{P})$, if $a \in F(\boldsymbol{P}_{-b})$ and $Margin_{\boldsymbol{P}}(a, b) > 0$, then $a \in F(\boldsymbol{P})$.

	Split Cycle	Ranked Pairs	Beat Path	Mini-max	Copeland	GETCHA /GOCHA	Uncov. Set	Instant Runoff	Plurality
Immunity to Spoilers	✓	–	–	✓	✓	✓	✓	–	–
Immunity to Stealers	✓	✓*	–	–	–	✓	✓	–	–
Stability for Winners	✓	–	–	–	–	✓	✓	–	–
Expansion Consistency	✓	–	–	–	–	✓/–	✓†	–	–

W. Holliday and EP. *Split Cycle: A New Condorcet Consistent Voting Method Independent of Clones and Immune to Spoilers*. https://arxiv.org/abs/2004.02350, 2021.

Distinguishing Split Cycle from other definitions of defeat

Split Cycle is distinguished from other definitions of defeat by:

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Positive Involvement

Definition

A VCCR satisfies **Positive Involvement in Defeat** if for any profile $\boldsymbol{P}$ and $a, b \in X(\boldsymbol{P})$, if a is not defeated by b in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by adding one new voter who ranks a above b, then a is still not defeated by b in $\boldsymbol{P}'$.

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Proposition

Split Cycle satisfies Positive Involvement in Defeat.

Key idea: Unequivocal increase in support for a candidate should not result in that candidate going from being a winner to being a loser.

1. *monotonicity*: if a candidate x is a winner given a preference profile $\boldsymbol{P}$, and $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by one voter moving x up in their ranking, then x should still be a winner given $\boldsymbol{P}'$.
   (fixed-electorate axiom)

2. *positive involvement*: if a candidate x is a winner given $\boldsymbol{P}$, and $\boldsymbol{P}^*$ is obtained from $\boldsymbol{P}$ by adding a new voter who ranks x in first place, then x should still be a winner given $\boldsymbol{P}^*$.
   (variable-electorate axiom)

No Show Paradox

The term "No Show Paradox" was introduced by Fishburn and Brams for violations of what is now called *negative involvement*: Adding a new voter who ranks a candidate last should not result in the candidate going from being a loser to a winner.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine, 56(4), pp. 207 - 214, 1983.

D. Saari. *Basic Geometry of Voting*. Springer, 1995.

No Show Paradox

Moulin changed the meaning of "No Show Paradox" to refer to violations of participation: A resolute voting method satisfies participation if adding a new voter who ranks x above y cannot result in a change from x being the unique winner to y being the unique winner.

H. Moulin. *Condorcet's Principle Implies the No Show Paradox*. Journal of Economic Theory 45(1), pp. 53 - 64, 1988.

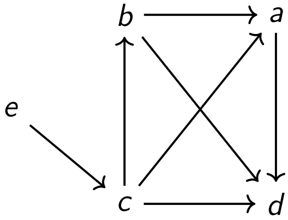
No Show Paradox

Peréz concludes that the Strong No Show Paradox is a common flaw of many *Condorcet consistent* voting methods, which are methods that always pick a Condorcet winner—a candidate who is majority preferred to every other candidate—if one exists.

J. Pérez. *The Strong No Show Paradoxes are a common flaw in Condorcet voting correspondences*. Social Choice and Welfare 18(3), pp. 601 - 616, 2001.

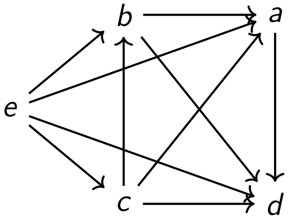
Violating Positive Involvement: Copeland

2	1	1
e	c	a
c	b	d
b	a	b
a	d	e
d	e	c



Copeland winners: $\{c\}$

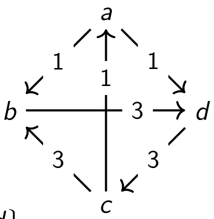
2	1	1	
e	c	a	c
c	b	d	e
b	a	b	d
a	d	e	c
d	e	c	a



Copeland winners: $\{e\}$

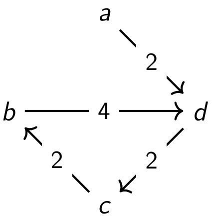
Violating Positive Involvement: Beat Path

1	1	1	1	2	1	1	1	1	1
a	d	c	c	a	b	a	d	d	b
d	c	a	b	c	d	b	c	b	a
c	b	b	d	b	c	d	a	c	d
b	a	d	a	d	a	c	b	a	c



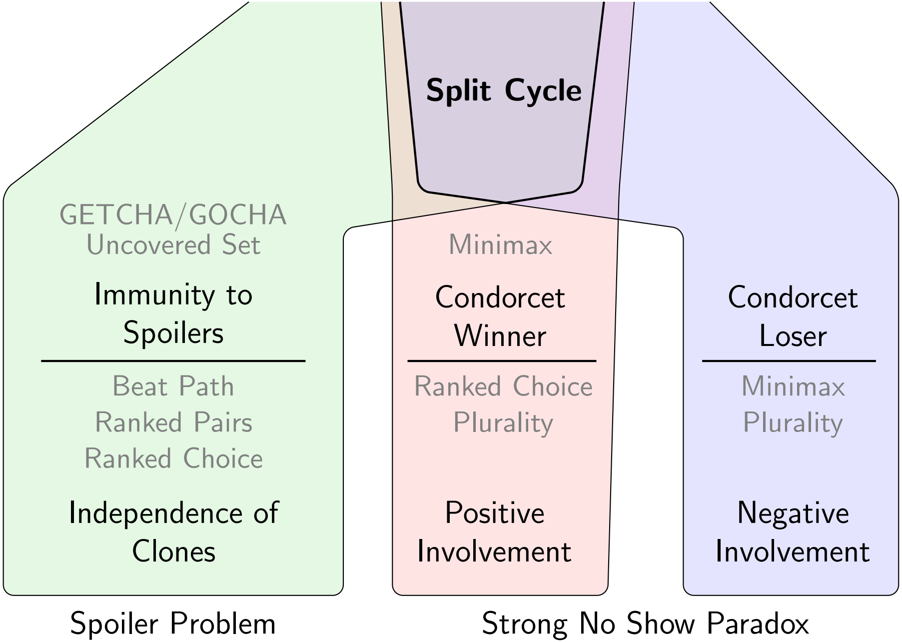
Beat Path winners: $\{a, b, c, d\}$

1	1	1	1	2	1	1	1	1	1	1
a	d	c	c	a	b	a	d	d	b	b
d	c	a	b	c	d	b	c	b	a	a
c	b	b	d	b	c	d	a	c	d	c
b	a	d	a	d	a	c	b	a	c	d



Beat Path winners: $\{a\}$

	Split Cycle	Ranked Pairs	Beat Path	Mini-max	Copeland	GETCHA /GOCHA	Uncov. Set	Instant Runoff	Plurality
Monotonicity	✓	✓	✓	✓	✓	✓	✓	–	✓
Positive Involvement	✓	–	–	✓	–	–	–	✓	✓
Negative Involvement	✓	–	–	✓	–	–	–	–	✓


Split Cycle
GETCHA/GOCHA
Uncovered Set
Immunity to Spoilers
Beat Path
Ranked Pairs
Ranked Choice
Independence of Clones
Minimax
Condorcet Winner
Ranked Choice
Plurality
Positive Involvement
Condorcet Loser
Minimax
Plurality
Negative Involvement
Spoiler Problem
Strong No Show Paradox

Axiomatic characterization

Beyond finding some axioms that distinguish Split Cycle from other proposed VCCRs, we sought a complete axiomatic characterization of Split Cycle.

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In "Axioms for Defeat in Democratic Elections," we characterize Split Cycle as

- the most resolute VCCR satisfying five standard axioms plus a weakening of Arrow's axiom of IIA that we call **Coherent IIA**.

Five standard axioms

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A1. Anonymity and Neutrality: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by swapping the ballots assigned to two voters, then x still defeats y in $\boldsymbol{P}'$ (Anonymity);

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A2. Availability: for every $\boldsymbol{P}$, there is some undefeated candidate in $\boldsymbol{P}$.

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Characterization with Coherent IIA

Given a class C of VCCRs and $g \in C$, we say that

$$g \text{ is the } \textit{most resolute} \text{ VCCR } \textit{in } C$$

if for every $f \in C$, profile $\boldsymbol{P}$, and $x, y \in X(\boldsymbol{P})$, if x defeats y in $\boldsymbol{P}$ according to f, then x defeats y in $\boldsymbol{P}$ according to g.

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See "Axioms for Defeat in Democratic Elections," *Journal of Theoretical Politics* (arXiv:2008.08451 [econ.TH]).

Definition

A VCCR $\boldsymbol{F}$ satisfies **Coherent Defeat** iff for any x and y that are not in any majority cycle, x defeats y iff $Margin_{\boldsymbol{P}}(x, y) > 0$.

Theorem (Ding, Holliday and EP 2022)

Split Cycle is the only VCCR satisfying A1-A5, Coherent IIA, Coherent Defeat, *and* Positive Involvement in Defeat.

Y. Ding, W. Holliday and EP (2022). *An Axiomatic Characterization of Split Cycle*. manuscript.

Distinguishing Split Cycle from other definitions of defeat

Split Cycle is distinguished from other definitions of defeat by:

1. responding in a reasonable way to **new candidates** joining the election;
   $\Rightarrow$ Stability for Winners
2. responding in a reasonable way to **new voters** joining the election.
   $\Rightarrow$ Positive Involvement

A problem for this optimistic story?

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Proposition

Split Cycle is not asymptotically resolvable.

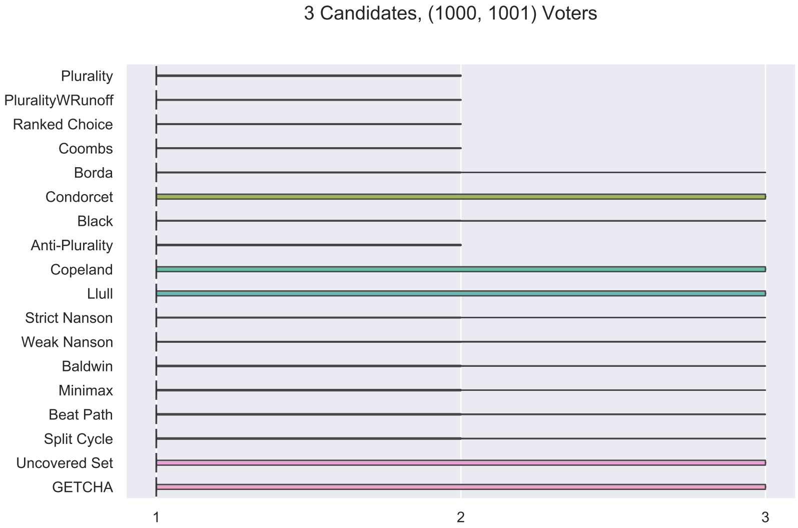
Violations of Quasi-Resoluteness

The known methods that satisfy Binary Expansion violate Asymptotic Resolvability/Quasi-Resoluteness.

Voting Method	3	4	5	6	7	8	9	10	20	30
Split Cycle	1	1.01	1.03	1.06	1.08	1.11	1.14	1.16	1.42	1.62
Uncovered Set	1.17	1.35	1.53	1.71	1.9	2.09	2.26	2.46	4.56	6.82
Top Cycle	1.17	1.44	1.8	2.21	2.72	3.31	3.94	4.68	13.55	22.94

Figure: Estimated **average sizes of winning sets** for profiles with a given number of candidates (top row) in the limit as the number of voters goes to infinity, obtained using the Monte Carlo simulation technique in M. Harrison Trainor, "An Analysis of Random Elections with Large Numbers of Voters," arXiv:2009.02979.

Sizes of Winning Sets



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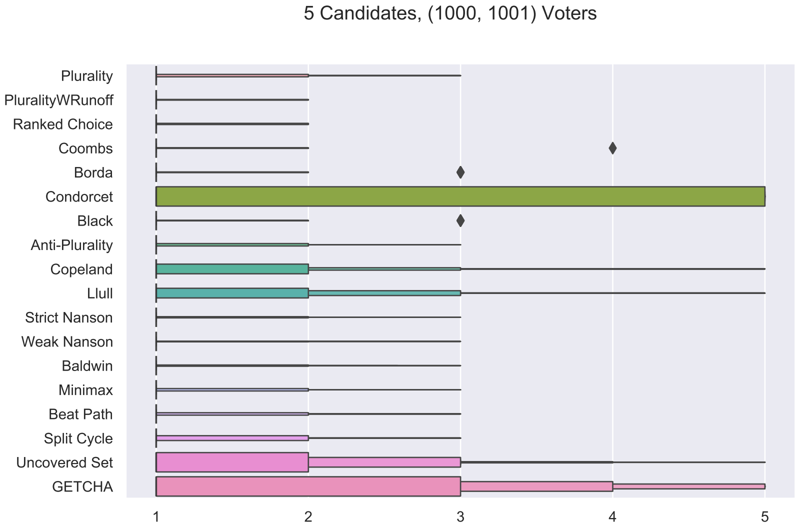

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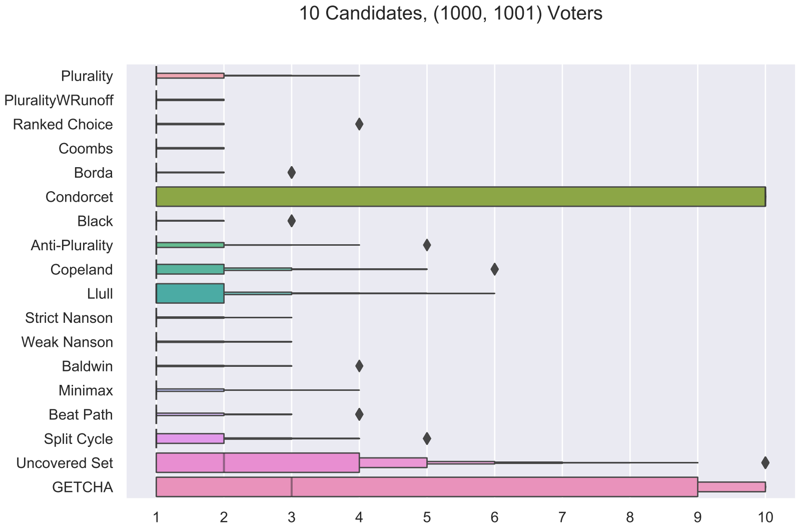

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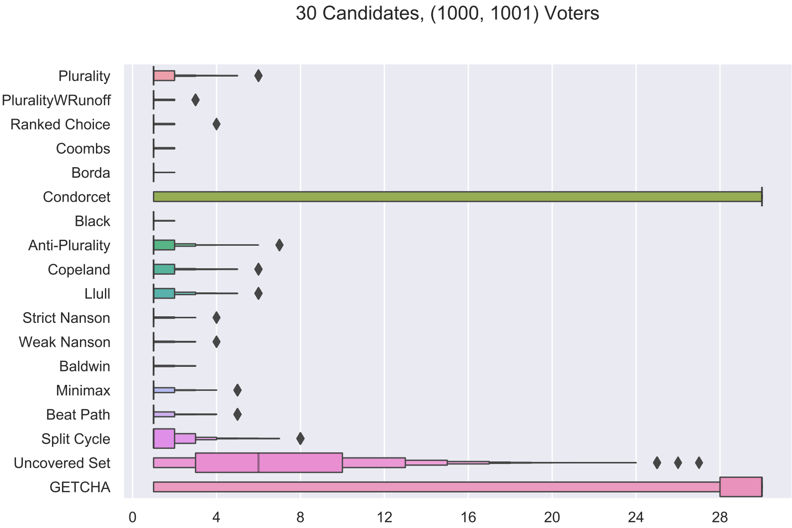

The Cost of Quasi-Resoluteness

Theorem (W. Holliday, EP, and S. Zahedian)

There is no Anonymous and Neutral voting method that satisfies Binary Expansion and Quasi-Resoluteness.

Moral: Making room for tiebreaking (runoff, lottery, etc.) is necessary and sufficient to find voting methods that satisfy Binary Expansion.

Multiple claims based on stability

The basic problem is that inevitably there are profiles with *multiple* candidates who have the same kind of claim to winning based on stability for winners:

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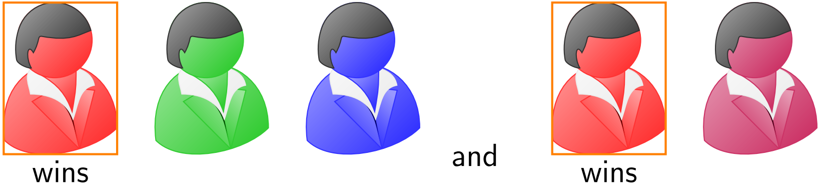

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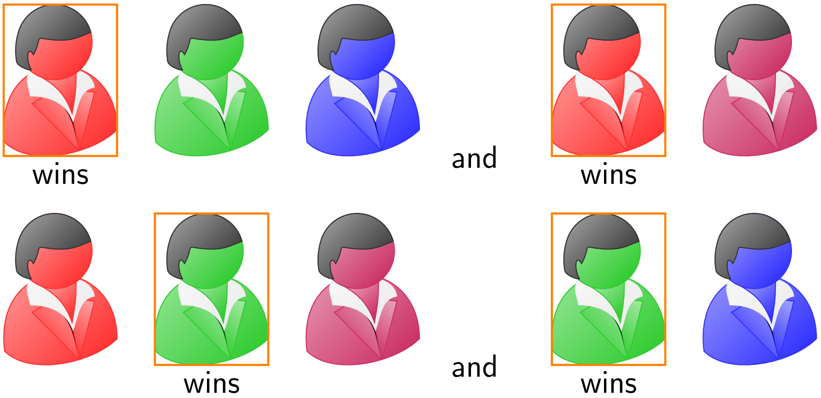

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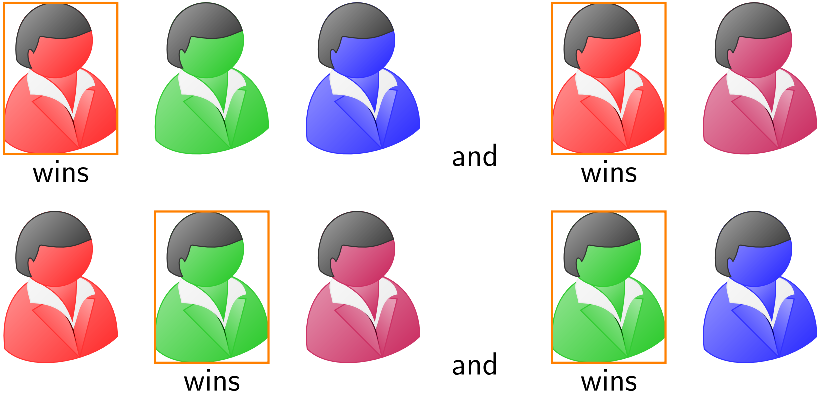
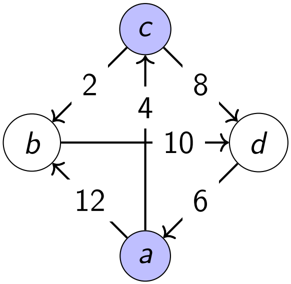


In such a situation—and only such a situation—it is legitimate to violate stability for winners for one of red or green in the name of tiebreaking between them.

Condorcetian candidates

Definition

Given a voting method F, profile $\boldsymbol{P}$, and $a \in X(\boldsymbol{P})$, we say that *a is Condorcetian for F in* $\boldsymbol{P}$ if there is some $b \in X(\boldsymbol{P})$ such that $a \in F(\boldsymbol{P}_{-b})$ and $Margin_{\boldsymbol{P}}(a, b) > 0$.



- There are two Condorcetian candidates a and c
- Beat Path elects c
- Ranked Pairs elects a

Stability for Winners with Tiebreaking

Definition

A voting method satisfies **Stability for Winners with Tiebreaking** if for any profile $\mathbf{P}$ and $a, b \in X(\mathbf{P})$, if a wins in $\mathbf{P}_{-b}$ and $Margin_{\mathbf{P}}(a, b) > 0$,

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- a wins in $\mathbf{P}$ or
- there are $a', b' \in X(\mathbf{P})$ such that a' wins in $\mathbf{P}_{-b'}$, $Margin_{\mathbf{P}}(a', b') > 0$, and a' wins in $\mathbf{P}'$.

That is, all winners are Condorcetian.

Recursion to the Rescue: Stable Voting

Our proposed voting method is Stable Voting, defined *recursively* as follows:

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  Find the first match such that $\boldsymbol{a}$ wins according to Stable Voting *after $\boldsymbol{b}$ is removed from all ballots*; this $\boldsymbol{a}$ is the winner for the original set of ballots.

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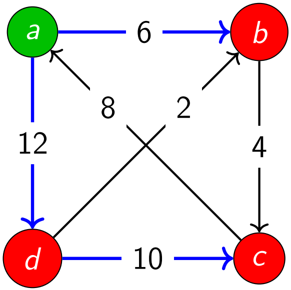
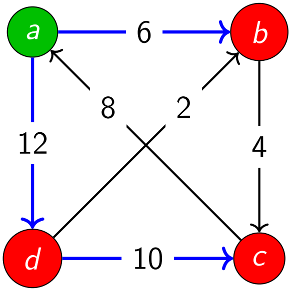
W. Holliday and EP. *Stable Voting*. arXiv:2108.00542 [econ.TH].



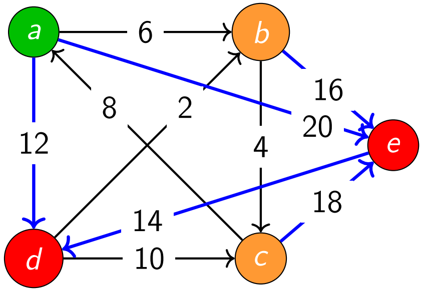
Stable Voting winner: a

Beat Path winner: a

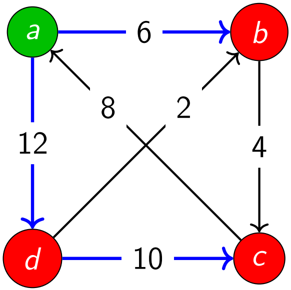
Ranked Pairs winner: a



Stable Voting winner: *a*
Beat Path winner: *a*
Ranked Pairs winner: *a*



Stable Voting winner: *a*
Beat Path winner: *b*
Ranked Pairs winner: *c*



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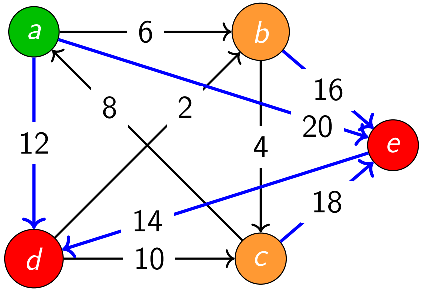

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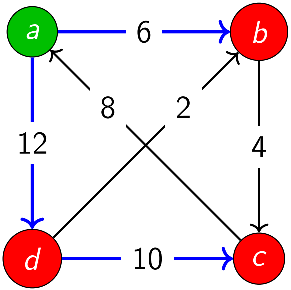
On the right, SV chooses the winner by going down the list of matches:



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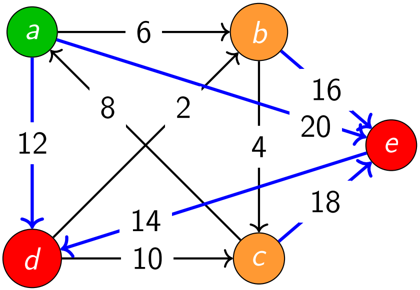

Stable Voting winner: a
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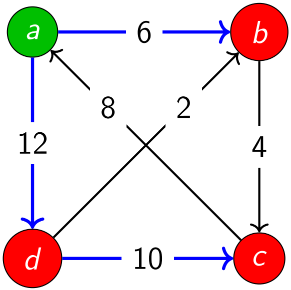
On the right, SV chooses the winner by going down the list of matches:

- a vs. e: margin of 20.



Stable Voting winner: *a*
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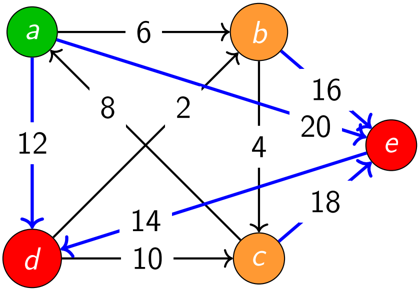

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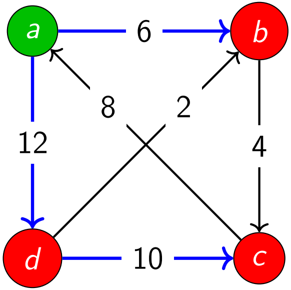
  a wins after removing ***e***. Hence ***a* is elected**.



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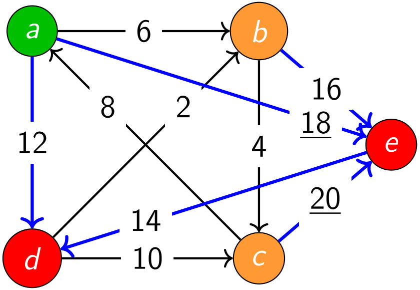

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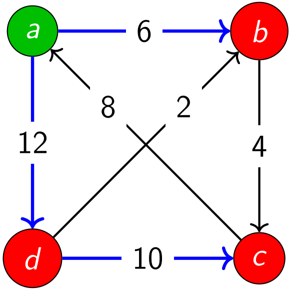

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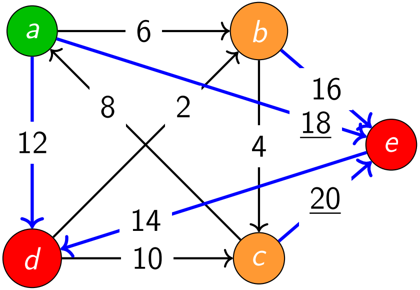

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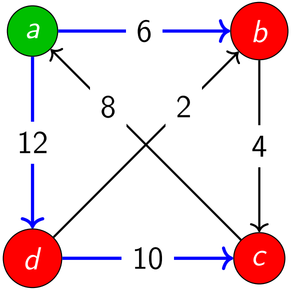

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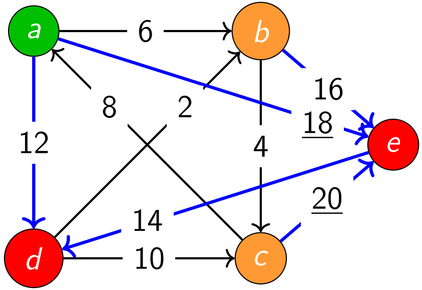

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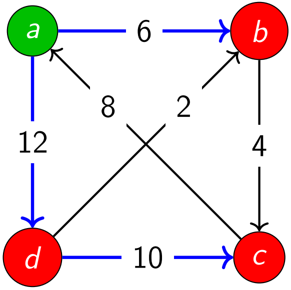

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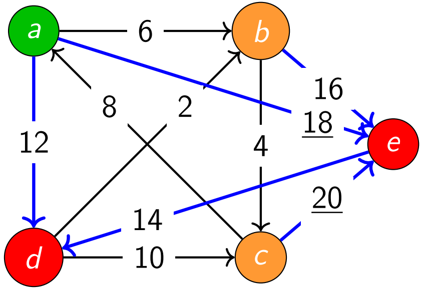

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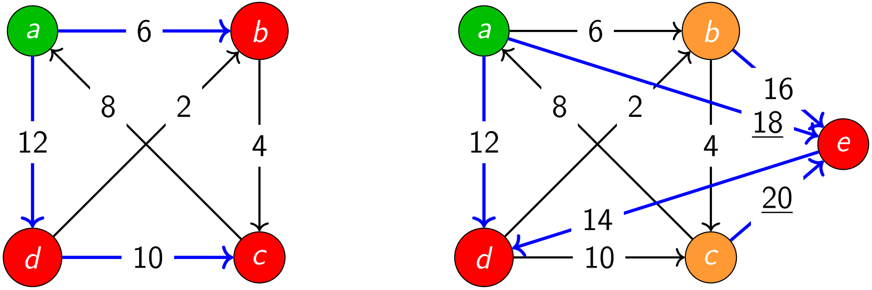
  c loses after removing e. Continue to next match:



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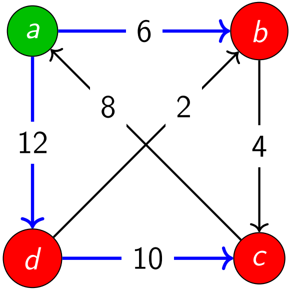

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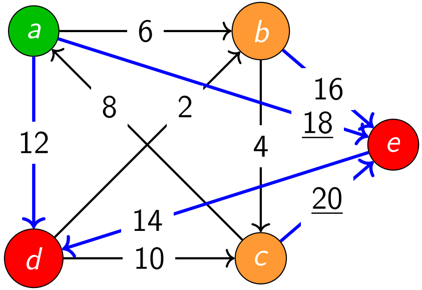

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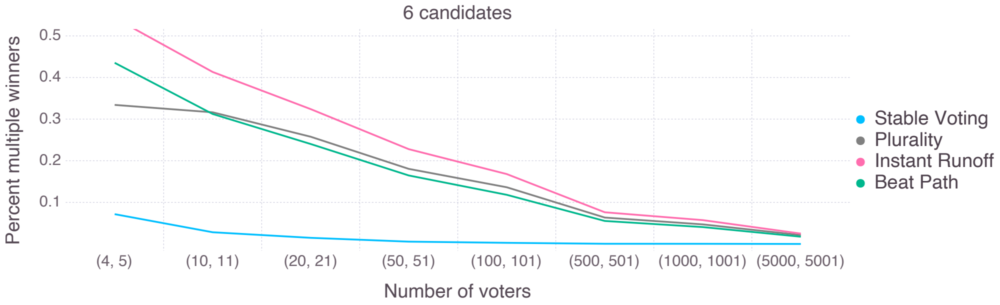
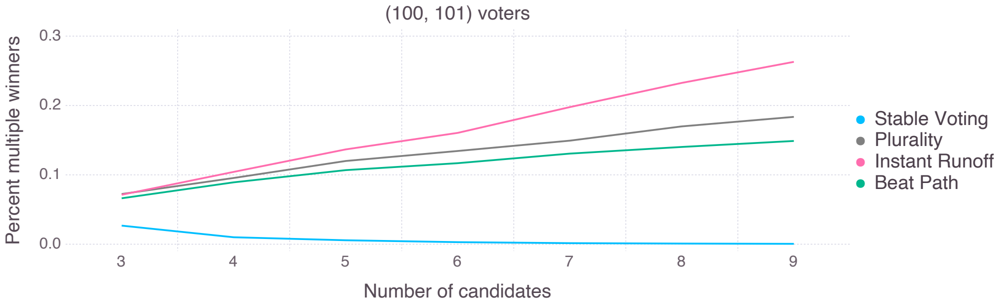
In fact, SV has a remarkable ability to avoid ties even in elections with small numbers of voters that can produce tied margins.


6 candidates
Percent multiple winners
0.5
0.4
0.3
0.2
0.1
(4, 5)
(10, 11)
(20, 21)
(50, 51)
(100, 101)
(500, 501)
(1000, 1001)
(5000, 5001)
Number of voters
Stable Voting
Plurality
Instant Runoff
Beat Path


(100, 101) voters
Percent multiple winners
0.3
0.2
0.1
0.0
3
4
5
6
7
8
9
Number of candidates
Stable Voting
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Beat Path

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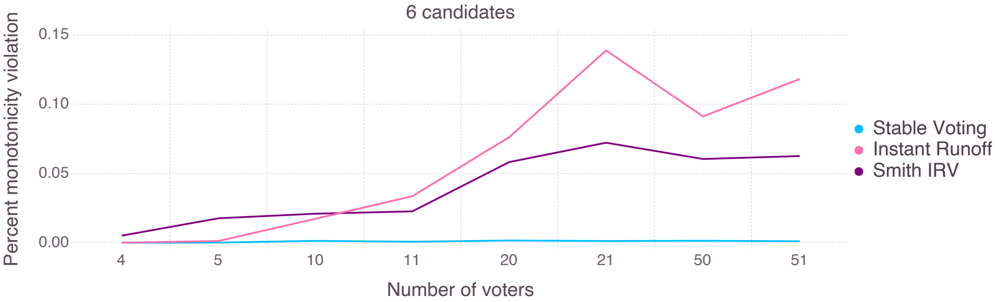
Re 2, the frequency with which Stable Voting violates monotonicity is minuscule compared to the frequency for Instant Runoff (in use in the Bay Area and NYC).

Monotonicity



Demo

stablevoting.org