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ACCEPTABLE SOCIAL CHOICE LOTTERIES*

1. Introduction

Lotteries have been used at least since biblical times to make certain types of social decisions. Thanks in part to the paper by Zeckhauser [29], which notes that "Social decision procedures usually do not allow lotteries on alternatives to compete as potential social choices", interest in the analysis of social choice lotteries has increased in the past few years. As far as I am aware there has not however been an attempt to identify the characteristics of situations in which a social choice lottery can be acceptable by current standards. The first main purpose of this paper will therefore be to suggest a set of characteristics for a social choice process that delineates minimal standards of acceptability for the use of a social choice lottery. At the same time I shall identify reasons why such lotteries are not used more often at the present time.

The second half of the paper then examines aspects of acceptable situations for social choice lotteries from a mathematical viewpoint. We shall look at these situations from the perspective of the preferences of the agents who have a direct stake in them. Our two main concerns will be the existence of a Condorcet alternative—one that is preferred by a majority of non-indifferent agents to every other alternative—and the nature of Pareto lotteries, which are lotteries that are not unanimously less preferred than other lotteries. In addition, the role of a compromise alternative will be considered.

Further introduction is provided in the next section which explains the basic formulation for social choice lotteries and briefly reviews the literature on the topic of social choice lotteries.

2. Social choice lotteries

Let X be a nonempty set of mutually exclusive decision alternatives that are viewed as the basic objects of choice in a group decision process. A *lottery* on X is a simple probability distribution on X so that if p is a lottery on X then $p(x)=0$ for all but a finite number of $x \in X$

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and $\sum p(x) = 1$ for those x that have $p(x) > 0$. Lotteries can be—and sometimes are—used to select a basic alternative as the social choice in a group decision process, and in this capacity we refer to them as social choice lotteries. If lottery p is used to make the social choice, then a random device that assigns probability $p(x)$ to each x for which $p(x) > 0$ is actuated to determine a 'winner'. Random devices used for this purpose range from coins and dice to somewhat more elaborate mechanisms used, for example, in draft lotteries and state or national lotteries.

Although elements in X may present aspects of risk or uncertainty to the individuals in the group, as when each $x \in X$ is a risky investment opportunity [10] or a political candidate whose positions on salient issues are ambiguous [22], the probabilities used in the social choice lotteries need not refer directly to these risks or uncertainties. For example, a political candidate may find it advantageous to be vague about his views on the issues (as in the 1976 election for President of the United States?), thus in effect creating a lottery over issue positions in the mind of the public. However, a social choice lottery to select a winning candidate in this case does not refer directly to lotteries over issue positions although the latter may affect the vote which, in turn, may affect the social choice lottery that is used. We know of course that social choice lotteries are seldom if ever used to choose winners of public elections and will give reasons for this in the next section.

On the other hand, there is a sense in which choice by lottery may be relevant in a direct majority-winner vote between candidates. In particular, if issue positions are interpreted as the basic alternatives, if each candidate in effect presents a lottery over issue positions to the electorate, and if the winner's actual positions and resultant policies are viewed as being selected according to his or her issue-position lottery, then the vote between the candidates can be taken to be a vote between social choice lotteries on issue positions. I shall not however focus on this type of interpretation in what follows since the social choice lotteries that will be considered are lotteries that are explicitly carried out by some random device as described earlier.

In discussing social choice lotteries I shall follow the precedent of virtually all the literature on the subject by concentrating on a generic subset A of X which may be thought of as the set of available alternatives or the admissible agenda that obtains in the situation at hand. The question of how available versus unavailable or infeasible alternatives are determined will not be addressed explicitly. With A the feasible set of alternatives, the social choice must be made from A; hence a social choice lottery will be admissible in the A context only if $p(x)=0$ for each $x \notin A$. Only admissible lotteries will be used in what follows. To the extent that the social choice from A is to be based on the attitudes, beliefs, values and perhaps votes of individuals, we presume in the spirit of Arrow's independence condition [1, 9, 25] that the only information from individuals about alternatives that will be used in the choice process is information about the feasible alternatives in A. This information may involve ballots and/or aspects of individuals' preferences on the single elements in A or on subsets of A or on lotteries on A.

An example will clarify several aspects of this formulation. We suppose that a panel or jury of n judges $(i=1, \ldots, n)$ is to award a 'prize' to one of m contestants $(j=1, \ldots, m)$ who have qualified for a certain competition such as a beauty contest, primary election, jury trial, or sports tournament. We shall let a_j denote the decision to award the 'prize' to contestant j. Hence the feasible set is $A=\{a_1, \ldots, a_m\}$. There is of course a huge number of balloting-scoring processes that are or could be used to make the choice in this type of setting. I shall mention several of these that involve social choice lotteries.

One procedure, discussed by Intriligator [19] and others [11, 24], would have each judge submit a ballot that amounts to a lottery on A. The probability $p^i(a_j)$ that judge i assigns to a_j might be interpreted as judge i's subjective probability that contestant j is the best contestant or the one who most deserves the prize. Social choice lotteries can be formed from the n lotteries $(p^1, \ldots, p^n)$ of the judges in a number of ways. One of these is Intriligator's average rule that defines $p(a_j)$ for the social choice lottery p by $p(a_j)=[p^1(a_j)+\ldots+p^n(a_j)]/n$. In this case $p(a_j)>0$ so long as $p^i(a_j)>0$ for some i. A different procedure sets $p(a_j)=1/N$ for each of the N values of j that maximizes $\sum_i p^i(a_j)$ and sets $p(a_j)=0$ for the others. This procedure will often result in a degenerate social choice lottery ($p(a_j)=1$ for some j) and involves only even-chance social choice lotteries on subsets of A.

Another approach to balloting asks each judge to vote for one contestant. With n_j the number of votes obtained by j and $n_1+n_2+\ldots+n_m=n$, the social choice proportional lottery rule takes $p(a_j)=n_j/n$ for each j. This rule has been discussed by Coleman [4] and Fishburn and Gehrlein [13, 14] in the $m=2$ setting. An alternative is to set $p(a_j)=1/N$ for each of the N values of j that maximizes n_j and to set $p(a_j)=0$ for the others. When $m=2$, the latter rule is the simple majority rule with a tie ($n_1=n_2$) broken by the flip of a fair coin [13, 14]. Obvious modifications of these procedures arise when each judge is allowed to vote for as many contestants as he pleases.

Other balloting methods ask the judges to rank some or all of the contestants from most preferred to least preferred, or to assign scores to each contestant, or to vote between contestants or social choice lotteries on contestants two at a time. Several authors [7, 8, 26, 27, 29] investigate the existence of a Pareto optimal social choice lottery and the existence of a simple majority equilibrium lottery when voters are presumed to have preference orders over lotteries. Others [2, 17, 20, 30] consider aspects of strategic (manipulative) voting when even-chance lotteries or more general types of lotteries compete for the social choice. An approximate conclusion of most of this work indicates that the potential use of social choice lotteries does little to alleviate, and indeed may aggravate, the problems that can arise in social choice theories where lotteries are not permitted. Nevertheless, there are situations in which lotteries seem quite natural and are used as a matter of course in arriving at social choices. One class of such situations will be developed in the next section and examined more closely in the final section.

3. Acceptable lotteries

One of the striking features of many social choice procedures is the lengths to which they will go to avoid the use of a nondegenerate social choice lottery in helping to determine the outcome of the process. In those instances in which balloting is used and the balloting-scoring procedure may yield a tie among two or more alternatives in A, a sequence of ballots, some of which may involve new sets of judges, is often used until a single winner is obtained. Selection of party candidates by national conventions to nominate presidential contenders in the United States provides one example. And if our Electoral College cannot reach a conclusion on who should be the next President, the matter goes to the U.S. House of Representatives for resolution. In a very elaborate and to the best of my knowledge unused election system involving multiple candidates [18, pp. 496–505], fourteen main steps for determining a winner are followed by ten additional steps for resolving ties, the last of which says that [p. 500]

> if all pairs of remaining undefeated candidates have been compared and found to be tied, declare that undefeated candidate elected who received most first choices. If two or more of them received the same highest number of first choices, declare elected that one of the candidates tied with most first choices who received on all the ballots most second choices. If there is a further tie, decide it by referring to third choices, and so on. If two or more candidates remain tied after the examination of all choices, declare one of them elected by lot.

In this system an even-chance lottery among a subset of candidates is to be used only as a last resort.

In addition to multiple ballots and special voting rules, odd numbers of judges or committee members and tie-breaking chairmen are often used to avoid deadlocks. In most jury trials which end in a hung jury a mistrial is declared when the issue might have been decided by some type of lottery. The aversion to justice by chance is well illustrated by a June 20, 1976 article in the New York Times (p. 11) which reported that the Louisiana Judiciary Commission recommended disciplinary action against a Baton Rouge city judge who gave the appearance of deciding cases by tossing coins.

Modern attitudes towards social choice lotteries have been shaped in large measure by Enlightenment thought that led to the doctrine of the freedom and moral responsibility of men to shape and control their destiny coupled with the doctrine of egalitarianism or the belief in human equality in social, political and economic spheres. Together, these principles led to the overthrow of theocracies and the divine right of kings and ushered in modern forms of participatory democracies.

According to the freedom-responsibility doctrine, the use of a lottery to make a social decision subverts man's control over his own affairs, denies his proper role as a moral agent responsible for the health and improvement of the social organism, and otherwise constitutes a step backwards into the dark ages by relegating the decision to blind chance. An historically interesting contrast to this viewpoint is provided by two verses from the book of Proverbs (Revised Standard Version): The lot puts an end to disputes and decides between powerful contenders (18:18); The lot is cast into the lap, but the decision is wholly from the Lord (16:33). Thus the outcome of the die cast by man to settle an issue reveals God's will.

Although the will of God has given way to the will of the people or the will of the majority in many societies, as the hand of God in matters of chance has been replaced by lady luck, certain social choice lotteries are sanctioned today. The coin flip before the kickoff at an American football game has become a secular ritual. The Irish sweepstakes, English football pools, and state lotteries are officially condoned money-raising activities. Lotteries are commonly used to select people for potential jury duty. The order of names on ballots is often determined by chance. And several years ago the United States instituted a draft lottery to correct inequities in its previous conscription system although the lottery has since been replaced by a volunteer system.

An examination of these situations reveals a number of common characteristics. First, there is a set of two or more qualifying agents, such as teams, ticket holders, or citizens or residents of a certain jurisdiction who meet prescribed criteria. Second, there is a 'prize' or a set of similar prizes (kickoff option, money, invitations to jury duty, positions on the ballot) to be awarded to the qualifying agents. Third, all agents have more or less uniform attitudes towards the desirability of each prize: both teams would like the kickoff option, everyone would like to win the sweepstakes or have his name first on the ballot, and most potential draftees would probably prefer not to be drafted. Fourth, each agent would acknowledge that all agents have a more or less equal claim on or right to each prize. And fifth, the agents do not actively compete to convince other agents or interested parties that they are more deserving of or better qualified to be awarded the prize(s). Since I know of no situation in which prizes are awarded by a sanctioned social choice lottery that does not have these characteristics, it is tempting to presume that they represent current minimal standards for acceptability of the use of a social choice lottery.

The fourth stipulation in the description of an acceptable situation, to the effect that each potential awardee feels that he has no more right to receive or avoid receiving a prize, than does his fellows, is essential. Thus a situation in which each claimant to a certain property believes that he has sole right to the property may be settled by a judge, or by a panel of judges or jurors, or perhaps by open warfare, but resolution by lottery is most improbable. Even if each member of a jury believed that the contestants had equal claims to the property, the jury is forbidden from reaching a verdict by lottery.

The fifth stipulation, regarding active competition, also affects court-case situations. There are other situations that satisfy the first four characteristics and fail on the fifth, thus eliminating them from the acceptable class. Suppose, for example, that a two-candidate political election situation satisfies our first four characteristics, with each candidate recognizing the claim of the other on the office at stake. Because the candidates actively try to convince the electorate of their qualifications, this situation does not satisfy the fifth characteristic. On the other hand, the sanctioned examples given earlier do satisfy this characteristic for the most part. In the jury selection and draft lottery cases, it is frequently held that all people who meet certain basic conditions have a duty to serve on a jury or to bear arms and, moreover, that each eligible person is able to do the required job. Hence there is no provision in the underlying philosophy of these situations for agents to compete actively for the positions although people sometimes do things (which others may find morally reprehensible) to disqualify themselves from consideration. Acts of this sort do of course challenge the passivity feature of the fifth characteristic to a degree.

We must be careful, however, in interpreting the fifth characteristic as a prerequisite for a social choice lottery since the use of a lottery may promote passivity which might not otherwise obtain. Thus if a present election-by-vote situation were changed to a social choice by lottery, active campaigning would probably cease. Despite this caution on the fifth characteristic, the situations given in our earlier examples appear, under prevailing social philosophy, to be the types in which active competition would not seem especially appropriate even if lotteries were not used.

The principle of equality is heavily involved with our earlier examples and characterization, especially with regard to the similar attitudes aspect of the third characteristic and the equal claim aspect of the fourth. When one prize is desired by all qualifying agents, when all recognize an equal claim to it, and when the award is not to be based on merit or superior qualifications, what fairer and more equitable way is there of awarding it than to use an even-chance lottery? The even-chance feature of the lotteries used in the examples also relates to people's ability to comprehend this type of lottery and to abide by its outcome in certain situations. On the other hand, people often have great difficulty understanding the probabilistic aspects of lotteries with unequal chances and may be quite averse to their use.

It is also of interest to note the interplay between the freedom-responsibility doctrine and the egalitarian doctrine in the types of situations under discussion. In the draft lottery and jury selection cases, the philosophical position noted above weakens the freedom-responsibility proscription against blind chance and leaves the way open for the egalitarian principle to sanction an even-chance lottery. On the other hand, consider the two-candidate political situation. A simple coin flip to determine a winner without any vote is ruled out not only by the freedom-responsibility rule but also by the egalitarian principle as it applies to potential voters. In this context the proportional lottery rule could be implemented by placing all marked ballots in a drum, drawing one ballot at random, and declaring the winner to be the candidate whose name is marked on the drawn ballot. (This rule is also pejoratively referred to as the 'random dictator' rule although it is similar to many of the even-chance lotteries in our sanctioned examples.) The proportional lottery rule is an even-chance lottery for voters since each voter has the same chance of naming the winner, and it clearly satisfies the egalitarian principle with regard to voters. There might however be some question of its fairness with respect to candidates. But regardless of this, the proportional lottery rule clearly clashes with the freedom-responsibility principle, which takes precedence in this situation and prescribes a nonrandom selection procedure. The egalitarian principle then enters the picture secondarily by prescribing a simple majority election in which each vote counts equally in a nonprobabilistic sense.

Despite various fairness arguments for the proportional lottery rule, both with respect to voters and minority candidates, I believe that its absence in election-type situations stems directly from the freedom-responsibility doctrine. Nevertheless, it may be remarked that a candidate's position on the ballot can affect the number of votes he gets, so that blind chance can affect the outcome in majority or plurality elections when ballot positions are determined randomly. More sophisticated ballots (as discussed in the next section) could correct this threat to the freedom-responsibility principle although these may be uneconomical in some cases and might be disliked by the candidates who might then be unable to advertise their positions on the ballot.

4. Mathematical aspects

The rest of this paper considers mathematical aspects of situations described by the characteristics for acceptable lotteries in the preceding section. Our purpose will be to examine these situations from the perspective of the preferences of the qualified agents. The following definitions will be useful.

An alternative x in the set A of basic feasible alternatives will be referred to as the *Condorcet alternative* [3, 5, 9, 12] if and only if more agents prefer x to y than prefer y to x for each $y \neq x$ in A. The *Condorcet principle*, whose congruence with the freedom-responsibility and egalitarian principles is often taken to be self-evident, says that the Condorcet alternative should be the social choice whenever there is such an alternative.

Alternative x is a *Pareto alternative* [1, 9] if and only if there is no y in A that every agent prefers to x. The *Pareto principle*, which is consistent with our other principles, says that the social choice should be a Pareto alternative.

The Condorcet and Pareto notions generalize in an obvious way to social choice lotteries on A. Thus lottery p is the *Condorcet lottery* if and only if more agents prefer p to q than prefer q to p for each lottery $q \neq p$. And p is a *Pareto lottery* if and only if there is no lottery q that every agent prefers to p. Since the even-chance lottery on A will receive special consideration we shall refer to it as p^*.

Under reasonable assumptions about agents' preferences we shall observe that A frequently has no Condorcet alternative and hence no Condorcet lottery [8]. In some cases all lotteries will be Pareto lotteries while in others very few lotteries will be Pareto lotteries. Compromise alternatives will also be considered. It will be noted that a compromise alternative that is a Condorcet alternative may fail to be a (degenerate) Pareto lottery.

4.1. *The Simple Paradox of Voting Example*

To set the stage for further discussion we shall begin with the situation in which three agents vie for a single prize. Let $A = \{a_1, a_2, a_3\}$ be the set of basic feasible alternatives with a_i the decision to award the prize to agent i. It will be assumed for the present that

Agent 1 prefers a_1 to a_2 to a_3
Agent 2 prefers a_2 to a_3 to a_1
Agent 3 prefers a_3 to a_1 to a_2.

Thus first-place preferences indicate that the prize is desired by each agent, and second and third-place preferences indicate that an agent still cares about who gets the prize if he does not. The preferences have of course been arranged to yield a majority cycle with no Condorcet alternative, since a_1 beats a_2 by a 2-to-1 majority, a_2 beats a_3, and a_3 beats a_1.

It will be assumed that each agent compares lotteries on A by the expected utility criterion. For convenience let 0 and 1 be each agent's utility for his least and most desired alternatives, respectively, and let $u_i \in (0, 1)$ be the von Neumann–Morgenstern utility [28] of agent i for his intermediate alternative. With $p_i = p(a_i)$, the expected utilities of the three agents for lottery p are

$$\begin{array}{ll} p_1 + p_2 u_1 & \text{for Agent 1} \\ p_2 + p_3 u_2 & \text{for Agent 2} \\ p_3 + p_1 u_3 & \text{for Agent 3.} \end{array}$$

Shepsle [26] notes that there is a lottery p that has a simple majority over each of the three basic alternatives if and only if its expected utility for each agent exceeds the agent's u_i value. However, when such a lottery exists there must be another lottery that has a simple majority over the first [8, 27]. Hence there can be no Condorcet lottery in this case. On the other hand, a simple argument shows that every lottery is a Pareto lottery. For with $d_i = p_i - q_i$ and $d_1 + d_2 + d_3 = 0$ for lotteries p and q, all agents will prefer p to q if and only if

$$\begin{array}{l} d_1 + d_2 u_1 > 0 \\ d_2 + d_3 u_2 > 0 \\ d_3 + d_1 u_3 > 0. \end{array}$$

If $d_i = 0$ for some i then, since $d_j + d_k = 0$ for the other two, one of the three inequalities must fail; and if $d_i \neq 0$ for all i then the inequalities and $\sum d_i = 0$ require $d_i > 0$ for exactly two of the three d_i, say d_1 and d_2 with $d_3 = -(d_1 + d_2)$, and in this case the third inequality is violated since $-(d_1 + d_2) + d_1 u_3 < 0$. Since neither the Condorcet principle nor the Pareto principle (applied either to basic alternatives or to lotteries) helps in any way to discriminate among lotteries, an even-chance lottery on A seems natural in this situation especially in view of the symmetry aspect of the basic preferences.

A further point of interest can be made in this context with regard to the possibility of a compromise prize. Suppose that the single prize could be split into three equal parts. Let c be the decision to award the compromise prize. The preceding preferences may then be supplemented as follows:

Agent 1. $a_1\, c\, a_2\, a_3$
Agent 2. $a_2\, c\, a_3\, a_1$
Agent 3. $a_3\, c\, a_1\, a_2$,

so that each agent prefers something to nothing. There are three nice features about c here: first, it is the Condorcet alternative in the context of $\{a_1, a_2, a_3, c\}$; second, the award of c seems reasonable and equitable; and third, the presence of c may ease our minds about the possibility of making the award by blind chance.

However, a closer look may reveal a disquieting possibility. In particular, every agent might prefer the even-chance lottery p^* on $\{a_1, a_2, a_3\}$ to the compromise alternative c. In other words, each would rather take his chances on getting the whole thing than to go for the three-way split. (People do not buy sweepstake tickets in the hope that their money will be refunded: they want a chance at the grand prize.) When this possibility obtains, the award of the Condorcet alternative c would not only thwart the will of the majority, it would go against the unanimous will of the group. Except perhaps for the aspect of chance, the choice of the even-chance lottery seems wholly consistent with the freedom-responsibility and egalitarian principles. On the other hand, one might argue that the Condorcet alternative c is the better choice since more agents will prefer c to the lottery outcome after the fact. If the latter position is taken, and if it is believed to be most in line with the two basic principles, then it seems necessary to demonstrate why the presumably morally responsible and rational agents would be ill-advised to implement an option that they uniformly prefer to the Condorcet alternative even though each knows that the risky option is more likely than not to leave him in a less preferred position.

4.2. *Ballot Positions*

We now turn to a multiple-prize situation to illustrate several aspects of formulation and analysis for this case. It will be assumed that three candidates, C_1, C_2 and C_3, compete in an election, the decision at the moment being the order of their names on the ballot. Let $C_iC_jC_k$ denote the ballot on which C_i's name is first, C_j's name is second, and C_k's name is last. The six ballot-position arrangements will be denoted as a_1 through a_6 where

$$\begin{aligned}
a_1 &= C_1C_2C_3\\
a_2 &= C_1C_3C_2\\
a_3 &= C_2C_1C_3\\
a_4 &= C_2C_3C_1\\
a_5 &= C_3C_1C_2\\
a_6 &= C_3C_2C_1.
\end{aligned}$$

Assuming that each candidate's preferences are governed solely by his position on the ballot, the following preference orders can be expected:

$$\begin{array}{ll}
C_1. & (a_1a_2)(a_3a_5)(a_4a_6)\\
C_2. & (a_3a_4)(a_1a_6)(a_2a_5)\\
C_3. & (a_5a_6)(a_2a_4)(a_1a_3).
\end{array}$$

Parentheses around alternatives indicate individual indifference between the alternatives; otherwise, preference decreases from left to right. There are two three-alternative simple majority cycles: a_1 beats a_5 beats a_4 beats a_1, and a_2 beats a_3 beats a_6 beats a_2. Other pairs of alternatives are tied. Hence, as in the preceding example, there is no Condorcet alternative.

With $p_i = p(a_i)$, the expected utilities of the candidates for lottery p on $A = \{a_1, \ldots, a_6\}$ are

$$\begin{array}{ll}
(p_1+p_2)+(p_3+p_5)u_1 & \text{for } C_1\\
(p_3+p_4)+(p_1+p_6)u_2 & \text{for } C_2\\
(p_5+p_6)+(p_2+p_4)u_3 & \text{for } C_3
\end{array}$$

where, as before, least and most preferred alternatives are assigned utilities of 0 and 1, and $u_i \in (0, 1)$ for each i. Unlike the preceding example, lotteries may fail to be Pareto lotteries in the present case. We consider first the even-chance lottery p^* on A that has $p_i^* = 1/6$ for each i.

THEOREM. *Lottery p^* is a Pareto lottery if and only if $u_1 = u_2 = u_3$.*

Proof. Lottery p^* is a Pareto lottery if and only if there does not exist a lottery $p = (p_1, \ldots, p_6)$ on A for which

$$(p_1 + p_2) + (p_3 + p_5)u_1 > \tfrac{1}{3} + u_1/3$$
$$(p_3 + p_4) + (p_1 + p_6)u_2 > \tfrac{1}{3} + u_2/3$$
$$(p_5 + p_6) + (p_2 + p_4)u_3 > \tfrac{1}{3} + u_3/3.$$

If $u_1 = u_2 = u_3$ then addition of these three inequalities gives $1 + u_1 + u_2 + u_3 > 1 + u_1 + u_2 + u_3$, which is impossible. On the other hand, suppose the u_i are not all equal and for definiteness assume that $u_1 > u_2$. Then it is easily verified that p with

$$p_1 = 0$$
$$p_2 = \tfrac{1}{3} - (u_1 + 3u_2)/12$$
$$p_3 = \tfrac{1}{6} + u_2/(6u_1)$$
$$p_4 = \tfrac{1}{6} - u_2/(6u_1) + (u_1 + 3u_2)/12$$
$$p_5 = \tfrac{1}{3}$$
$$p_6 = 0$$

is a lottery that satisfies the three preceding inequalities since substitution and reduction yields $u_1 > u_2$ in each inequality. This completes the proof of the theorem.

Since it would be unusual at the very least to attempt to determine the u_i in an actual ballot situation, and since the egalitarian principle might be interpreted as allowing only p^* as a potentially acceptable lottery, the preceding theorem might be taken to be little more than a technical curiosity. Nevertheless, it does suggest that in cases where more general lotteries might be considered, the 'obviously equitable' even-chance lottery may be unanimously less preferred than some other lottery.

When the equality condition on the u_i does not hold, many other lotteries will fail to be Pareto lotteries. In general, with $d_i = p_i - q_i$, every candidate will prefer lottery p to lottery q if and only if

$$(d_1+d_2)+(d_3+d_5)u_1>0$$
$$(d_3+d_4)+(d_1+d_6)u_2>0$$
$$(d_5+d_6)+(d_2+d_4)u_3>0.$$

As in the foregoing proof, suppose for definiteness that $u_1>u_2$. Then, with α_0, α_1 and α_2 positive and

$$d_2=-\alpha_1-d_1$$
$$d_3=-\alpha_0-d_1$$
$$d_4=\alpha_0+\alpha_1+d_1$$
$$d_5=\alpha_0+\alpha_2+d_1$$
$$d_6=-\alpha_0-\alpha_2-d_1,$$

so that $\sum d_i=0$, substitution in the preceding three inequalities shows that they all hold if and only if

$$\alpha_2 u_1>\alpha_1>(\alpha_0+\alpha_2)u_2.$$

Since $u_1>u_2$, it is always possible to select positive values of the α_j that satisfy this double inequality; moreover, if every q_i is positive, sufficiently small values of $|d_1|$, α_0, α_1 and α_2 can be chosen to guarantee that $p_i\geq 0$ for each i. Hence lottery q cannot be a Pareto lottery if $q_i>0$ for all i and the u_i are not all equal. In addition, if $u_1>u_2$, then by taking $d_1=-\alpha_0-\alpha_2$ or $d_1=-\alpha_0-\alpha_1$, it can be seen that q is not Pareto if either $q_1>0$ and $q_4>0$ or $q_1>0$ and $q_6>0$. On the other hand, if $q_1+q_2=1$, or $q_1+q_3=1$, or $q_5+q_6=1$, then q will be a Pareto lottery.

The following theorem summarizes part of the preceding discussion. It omits situations in which some of the p_i equal zero.

THEOREM. *If $p_i>0$ for $i=1,2,\ldots,6$, then lottery p is a Pareto lottery if and only if $u_1=u_2=u_3$.*

Hence the vast majority of lotteries will not be Pareto lotteries when the candidates do not have essentially similar risk attitudes towards their intermediate alternatives. If the u_i were actually estimated in such a case, then a reasonable lottery to use would be a maximin lottery. Such a lottery maximizes the minimum of the three agents' expected utilities and must be a Pareto lottery.

In the case at hand there is a natural compromise alternative c which directs that each of the six a_i orders appears on about one-sixth of the ballots. If every candidate prefers c to his intermediate alternatives, such as $(a_1a_2)c(a_3a_5)(a_4a_6)$ for C_1, then c will be the Condorcet alternative. But, as before, it could also be true that every candidate prefers p^* on $\{a_1, \ldots, a_6\}$ to c. However, p^* in this case will itself be unanimously less preferred than some other lottery if the u_i are not all equal.

4.3. *One Indivisible Prize*

It will be assumed henceforth that one indivisible and uniformly desired prize is to be awarded to one of n agents, with a_i the decision to award the prize to i. We consider first the situation in which each agent doesn't care who gets the prize if he doesn't get it. Using parentheses again to denote indifference, the preferences will be as follows:

$$\begin{array}{ll} \text{Agent } 1. & a_1(a_2a_3 \ldots a_n) \\ \text{Agent } 2. & a_2(a_1a_3 \ldots a_n) \\ \vdots & \\ \text{Agent } n. & a_n(a_1a_2 \ldots a_{n-1}). \end{array}$$

All alternatives are tied under binary comparisons since if $i \neq j$ then i prefers a_i to a_j, j prefers a_j to a_i, and the others are indifferent. Moreover, all lotteries on $A = \{a_1, \ldots, a_n\}$ are Pareto lotteries even though for every lottery there is another lottery that is preferred to the first by $n-1$ of the n voters: by reducing one $p_j > 0$ and increasing every p_i for $i \neq j$, we obtain a new lottery that everyone except j prefers to p.

The egalitarian principle supports the even-chance lottery p^*, but other arguments for p^* are also available. For example, p^* is the maximin lottery since it uniquely maximizes the minimum, over agents, of the ratio of the utility of the lottery minus the utility of the worst alternative to the utility of the best minus the utility of the worst. In addition, a little calculus shows that the proportion of the lottery space $\{(p_1, \ldots, p_n)$: $p_i \geq 0$ for all i and $\sum p_i = 1\}$ in which lottery p would be beaten by an $n-1$ to 1 vote is (for $n \geq 3$)

$$p_1^{n-1} + p_2^{n-1} + \ldots + p_n^{n-1}.$$

This proportion equals 1 if one of the $p_i = 1$, and it is minimized with value n^{-n+2} when $p_1 = p_2 = \ldots = p_n = 1/n$. Hence the unique lottery that is least likely to be defeated $n-1$ to 1 by another lottery chosen at random by the uniform distribution over the lottery space is p^*.

We now extend our analysis by assuming that each agent cares who gets the prize if he does not. To be more precise, let Case K denote the generic situation in which each agent ranks K ($<n$) of the a_i by decreasing preference with the one that awards him the prize in first place and in which the agent is indifferent among the remaining $n-K$ alternatives, which are least preferred. The two preceding paragraphs have examined $K=1$. With $n=5$, the preference orders $a_1a_4(a_2a_3a_5)$ and $a_1a_5a_2(a_3a_4)$ would be $K=2$ and $K=3$ orders respectively for agent 1. When $K \geq 2$, some lotteries, including p^*, can fail to be Pareto lotteries. I leave it to the reader to give an example of this when $(K, n) = (2, 3)$.

A Case K profile consists of one preference order for each agent in the Case K format. Since each agent can select and order the $K-1$ alternatives that immediately follow his most preferred alternative in $(n-1)(n-2)\ldots(n-K+1) = (n-1)!/(n-K)!$ ways, there are $[(n-1)!/(n-K)!]^n$ different Case K profiles for a given $n > K$. If $K > 1$ and $n \geq 3$ then some of these profiles will have Condorcet alternatives. For example, of the eight profiles for $(K, n) = (2, 3)$, six have Condorcet alternatives and two do not. One of the two that have no Condorcet alternative was displayed in the first example of this section.

Because each Case K profile has each alternative in first place in exactly one order, it seems likely that the proportion of Case K profiles that have no Condorcet alternative will exceed the proportion of profiles that have no Condorcet alternative when the first-place restriction is removed and each agent can have any alternative in first place. A profile of the latter type, in which each agent ranks K of the alternatives in decreasing preference with the other $n-K$ less preferred than the preceding K and indifferent to each other, will be called a Case K^* profile. There are $[n!/(n-K)!]^n$

different Case K^* profiles for $n > K$. Of the 216 Case K^* profiles for $(K, n) = (2, 3)$, 204 have Condorcet alternatives and 12 do not. Hence for $(K, n) = (2, 3)$, the proportion of Case K profiles with no Condorcet alternative is $2/8 = 0.25000$ and the proportion of Case K^* profiles with no Condorcet alternative is $12/216 = 0.05555\ldots$.

Table I compares the proportions of profiles that have no Condorcet alternative for the two cases for each relevant (K, n) pair up to $n = 5$. The table shows that the Case K proportion exceeds the Case K^* proportion in every instance. Hence the following conjecture seems appropriate.

CONJECTURE. *For every pair of integers (K, n) with $n > K \geq 1$, the proportion of Case K profiles that have no Condorcet alternative exceeds the proportion of Case K^* profiles that have no Condorcet alternative.*

This conjecture is obviously true for $K = 1$ but I have been unable to establish its status for any larger value of K. One way to approach it begins with the observation that the number of Case K^* profiles equals n^n times the number of Case K profiles. Hence if for each Case K profile in which a_i is the Condorcet alternative it is possible to identify at least n^n Case K^* profiles that have a_i as the Condorcet alternative in such a way that these do not overlap the n^n or more Case K^* profiles with a_i the Condorcet alternative identified for any other Case K profile in which a_i is the Condorcet alternative, then the conjecture is essentially true.

Readers who are familiar with other attempts to compute the likelihood of no Condorcet alternative [6, 15, 16, 21, 23] will readily recognize a connection between the foregoing conjecture and other work. In particular, if the conjecture is true for all (K, n) with $K = n - 1$, then it says that the likelihood of there being a Condorcet alternative when all voters independently select a linear preference order with equal probability $(1/n!)$ from the set of $n!$ linear orders on A (the 'impartial culture' case with the same number of voters as alternatives) will exceed the likelihood of there being a Condorcet alternative when voter i always has a_i in first place and selects the remainder of his linear preference order with equal probability $(1/(n-1)!)$ from the set of $(n-1)!$ linear orders on $\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$.

Table I and the associated conjecture suggest that the likelihood of no Condorcet alternative will be fairly significant in the type of

TABLE I
Proportions of Case K Profiles and Case K^* Profiles that have no Condorcet Alternative (no alternative that has a strict simple majority over every other alternative).

	$K=1$		$K=2$		$K=3$		$K=4$	
	Case K	Case K^*	Case K	Case K^*	Case K	Case K^*	Case K	Case K^*
$n=2$	1.00	0.50000						
$n=3$	1.00	0.22222	0.25000	0.05556				
$n=4$	1.00	0.23438	0.85185	0.59549	0.85185	0.65799		
$n=5$	1.00	0.32640	0.62891	0.43609	0.46470	0.29300	0.35185	0.19952

situation examined here. Hence if an attempt is made to resolve the situation by Condorcet's principle without the use of a social choice lottery, the attempt can easily fail. In as much as other reasons that require no explicit vote can sanction the use of an even-chance social choice lottery in the present context, it would probably be stretching the point to claim that the likely absence of a Condorcet alternative is a primary reason for the use of a lottery. Nevertheless it can hardly diminish the support for such usage.

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