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Lotteries and Social Choices*

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1. INTRODUCTION

Lotteries in social choice date at least from the first group which agreed to draw lots or flip a coin to settle an issue of concern to the group. They are by no means a modern invention: The lot puts an end to disputes and decides between powerful contenders [Proverbs 18:18. RSV].

More recently, thanks largely to the expected-utility theory of von Neumann and Morgenstern [28] after the fashion of Bernoulli [3], several theoretical excursions connecting chance and social choice have been made. One approach, pursued by Hildreth [17] and Harsanyi [14] and further examined by Vickrey [27], Diamond [8], Pattanaik [22] and Sen [25], among others, involves the use of Bernoullian utilities to determine a social choice on the basis of maximum sums of individual utilities. Adopting a somewhat different emphasis, Coleman [6] and Riker and Ordeshook [23] look at voter behavior and strategy on the basis of the voters' expected utilities.

However, most of this work does not presume that a lottery or chance device might actually be used in making the social choice. A clear exception to this is Zeckhauser's study [29, p. 69], which explores the relationship between simple majorities and lotteries and "demonstrates that unattractive social choices may result whenever lotteries are not allowed to compete. However, it also shows that whenever lotteries need to be considered as serious contenders, intransitivities in social choice will always arise. To avoid such difficulties, there must be a certain alternative that defeats not only all certain alternatives, but all lotteries as well".

To illustrate this, suppose that a three-member committee must select one of four certain alternatives from $\{a, b, c, d\}$ and that the members feel as follows:

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1. a is terrific; d is all right and is slightly better than b and c, which are satisfactory;

2. b is terrific; d is all right and is slightly better than a and c, which are satisfactory;

3. c is terrific; d is all right and is slightly better than a and b, which are satisfactory.

Thus d has a simple majority over each of a, b and c and would therefore be the choice under Condorcet's criterion [7; 4, p. 57] when lotteries do not compete. However, it so happens that each member prefers an even-chance lottery on $\{a, b, c\}$ to d; each would rather gamble for his favorite and settle the issue in that way. Hence the Condorcet alternative is Pareto-dominated by a lottery. This lottery is not itself dominated by another lottery although there are other lotteries (e.g., an even-chance lottery on $\{a, b\}$) that will be preferred to it by a majority of the committee. Indeed, no lottery will have a simple majority over every other lottery, provided that certain assumptions about individual preferences hold true. This may or may not be cause for alarm. My present feelings, which are related to the discussions in [11, 12], are that it is not as serious a matter as some would take it to be.

The present paper, motivated by the preceding example and by the work of Hausner [15], Aumann [1, 2], Kannai [20] and Fishburn [9, 13] in utility theory, considers lotteries in social choice under assumptions about individuals' preferences that are considerably weaker than the assumptions [9, 16, 21, 28] which imply the existence of a Bernoullian or von Neumann–Morgenstern utility function for each individual. We shall focus on two main aspects of social choice in this setting. The first is the existence of admissible or undominated lotteries. The second is the connection between simple majorities and lotteries, which supplements results of Zeckhauser [29] and Shepsle [26].

2. Summary of Results

To provide an indication of the technical developments in succeeding sections I shall summarize the main results of the paper.

Each individual is presumed to have a preference relation on the set of lotteries defined from a set of basic alternatives. The key notion of the paper is a form of Pareto dominance. We shall say that lottery x dominates lottery y if and only if each individual either prefers x to y or is "strongly indifferent" between x and y (for all z, x is indifferent to z if and only if

y is indifferent to z), and at least one individual prefers x to y. An admissible (Pareto optimal) lottery is a lottery that is not dominated by another lottery.

The weakest assumption that is used for an individual's preference relation on the lottery set says that this relation is transitive and irreflexive and that preference [strong indifference] between lotteries x and y implies preference [strong indifference] between the compound lotteries $\lambda x + (1 - \lambda)z$ and $\lambda y + (1 - \lambda)z$, $0 < \lambda < 1$. Theorem 1 shows that if this holds for each individual and if the set of basic alternatives is finite then the set of admissible lotteries contains a basic alternative and is describable in a fairly simple manner. Moreover, if the set of inadmissible lotteries is not empty then it is convex. Using the same individual assumption, we show later that the only type of lottery that can have a strict simple majority over every other lottery is a lottery that assigns probability one to a basic alternative (Theorem 3). In addition, if individual preferences on the basic alternatives are single peaked with a one-point peak for each individual, and if one basic alternative has a strict simple majority over every other basic alternative, then it is admissible in the lottery set (Theorem 4). This is in sharp contrast to the non-single-peaked example in the introduction where the Condorcet basic alternative d is inadmissible.

Several other results are derived when the individual preference assumptions of the preceding paragraph are supplemented with the condition that preference [strong indifference] between compound lotteries $\lambda x + (1 - \lambda)z$ and $\lambda y + (1 - \lambda)z$, $0 < \lambda < 1$, implies preference [strong indifference] between x and y. When these assumptions hold for each individual and the basic set is finite, we obtain the von Neumann–Morgenstern solution property which says that each inadmissible lottery is dominated by some admissible lottery (Theorem 2). An example preceding Theorem 2 shows that this can be false under the preference assumptions used for Theorem 1, where an inadmissible lottery might be dominated only by other inadmissible lotteries. The final theorem of the paper shows that if individual preferences satisfy slightly stronger conditions, including transitive indifference, if preferences on the basic alternatives are single peaked (with no restriction on the number of peak alternatives for each individual), and if one basic alternative has a strict simple majority over every other basic alternative, then it is admissible in the lottery set.

The next section discusses the individual preference assumptions. Admissibility is examined in Section 4, the notion of a convex choice set is mentioned briefly in Section 5, and simple majority is analyzed in Section 6.

3. Individual Assumptions

Throughout, B denotes the set $\{a, b, c,...\}$ of basic feasible alternatives in the situation at hand, and X denotes the set $\{x, y, z,...\}$ of all simple probability distributions on B. Each $x \in X$ is a function from B into $[0, 1]$ with $x(b) \geqslant 0$ for all $b \in B$, $\sum\{x(b): b \in B\} = 1$, and $\sum\{x(b): b \in A\} = 1$ for some *finite* subset $A \subseteq B$.

The elements in X are the obvious abstract forms of the lotteries that might be used to select a basic alternative. No notational distinction will be made between a basic alternative and the distribution that assigns probability 1 to it. That is, $b \in B$ is a basic alternative, and $b \in X$ is the probability distribution that assigns probability 1 to $b \in B$.

If $x, y \in X$ and if λ is a real number then $\lambda x + (1 - \lambda)y$ is the function from B into Re (the real numbers) for which $(\lambda x + (1 - \lambda)y)(b) = \lambda x(b) + (1 - \lambda)y(b)$ for all $b \in B$. If $0 \leqslant \lambda \leqslant 1$ then $\lambda x + (1 - \lambda)y \in X$. More general linear combinations are formed in the obvious way: If $x_k \in X$ and $\lambda_k \in \mathrm{Re}$ for $k = 1,..., K$, and if $\sum \lambda_k = 1$, then $\sum_k \lambda_k x_k$ is the function from B into Re for which $(\sum_k \lambda_k x_k)(b) = \sum \lambda_k x_k(b)$ for all $b \in B$. If $0 \leqslant \lambda_k \leqslant 1$ for all k then $\sum_k \lambda_k x_k \in X$.

A binary relation $\succ$ on X is a *strict partial order* if it is irreflexive (not $x \succ x$) and transitive ($x \succ y$ & $y \succ z \Rightarrow x \succ z$). We will generally assume that each individual's preference relation is a strict partial order: the preference relation for individual i ($i = 1,..., n$) will be written as $\succ_i$.

Several other relations are defined from $\succ$ as follows:

$$x \sim y \Leftrightarrow \text{not } (x \succ y) \text{ \& not } (y \succ x)$$

$$x \succcurlyeq y \Leftrightarrow x \succ y \quad \text{or} \quad x \sim y$$

$$x \approx y \Leftrightarrow (x \sim z \Leftrightarrow y \sim z, \text{ for all } z \in X)$$

$$x \gtrsim y \Leftrightarrow x \succ y \quad \text{or} \quad x \approx y.$$

In usual terminology, $\sim$ denotes indifference, and $\succcurlyeq$ is a preference-or-indifference relation. When $\succ$ is a strict partial order, $\sim$ may be intransitive, but $\approx$ is transitive and is therefore an equivalence. The relation $\approx$ might be thought of as a "strong indifference" relation. We shall call $\succ$ a *weak order* (in the strict sense) if and only if it is a strict partial order and $\sim$ is transitive: in this case, $\succcurlyeq$ is transitive and complete ($x \succcurlyeq y$ or $y \succcurlyeq x$, for all $x, y \in X$) and $\sim = \approx$, so that indifference is tantamount to strong indifference. Proofs of these things are given in Fishburn [9, Chap. 2].

The typical axioms for the lottery context assume that $\succ$ is a weak order in addition to other properties specified below in Definition 3.

Throughout most of this paper we shall use only the properties in the following definition.

Definition 1. A binary relation $\succ$ on X satisfies the weak individual axiom if and only if, for all $x, y, z \in X$ and all $\lambda \in (0, 1)$,

1. $\succ$ on X is a strict partial order,
2. $x \succ y \Rightarrow \lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z$, and
   $x \approx y \Rightarrow \lambda x + (1 - \lambda)z \approx \lambda y + (1 - \lambda)z$.

Part 2 of this definition states that when two distributions are diluted by a third in the same way, then $\succ$ or $\approx$ holds for the dilutions when $\succ$ or $\approx$ holds initially. From a psychological viewpoint, the $\approx$ part seems unexceptionable. But if λ is sufficiently near to zero then the two combinations in the $\succ$ part may be so overwhelmed by the dilution term $(1 - \lambda)z$ that they will be virtually indistinguishable and hence indifference might hold between them even though $x \succ y$. The normative argument in support of $x \succ y \Rightarrow \lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z$ follows the usual two-stage interpretation of the combinations, and need not be repeated here.

One of the results that we wish to obtain (Theorem 2) does not follow from the weak individual axiom. It requires the following stronger set of assumptions.

Definition 2. A binary relation $\succ$ on X satisfies the moderate individual axiom if and only if, for all $x, y, z \in X$ and $\lambda \in (0, 1)$,

1. $\succ$ on X is a strict partial order,
2. $x \succ y \Leftrightarrow \lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z$, and
   $x \approx y \Leftrightarrow \lambda x + (1 - \lambda)z \approx \lambda y + (1 - \lambda)z$.

This adds two antidilution statements to the weak individual axiom, namely $\lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z \Rightarrow x \succ y$, and $\lambda x + (1 - \lambda)z \approx \lambda y + (1 - \lambda)z \Rightarrow x \approx y$. The first of these simply says that if one distribution is preferred to another and if both have a "common" part, namely, $(1 - \lambda)z$, then the preference must be a result of their "different" parts. The second seems more vulnerable than its converse, since a λ near 0 might cause $\lambda x + (1 - \lambda)z \approx \lambda y + (1 - \lambda)z$ because of dilution even though $x \approx y$ is false.

Finally, for comparison and later illustrations, we mention a typical set of assumptions for von Neumann-Morgenstern expected utility.

Definition 3. A binary relation $\succ$ on X satisfies the strong individual axiom if and only if, for all $x, y, z \in X$ and all $\lambda \in (0, 1)$,

1. $\succ$ on X is a weak order,
2. $x \succ y \Rightarrow \lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z$,
3. $x \succ y \;\&\; y \succ z \Rightarrow \alpha x + (1 - \alpha)z \succ y$ and $y \succ \beta x + (1 - \beta)z$ for some $\alpha, \beta \in (0, 1)$.

Although it may appear that several aspects of the weak and moderate individual axioms are missing from Definition 3, the addition of the so-called Archimedean axiom (part 3) allows the derivation of the weak and moderate axioms from the strong axiom. This will be clear from Lemma 3 given below, proofs of which are given by Jensen [18] and Fishburn [9, Chap. 8]. Proofs of Lemmas 1 and 2 are given in the Appendix of this paper.

LEMMA 1. *Suppose that $\succ$ on X satisfies the weak individual axiom; that $x_k, y_k \in X$, $x_k \succsim y_k$ and $\lambda_k \geqslant 0$ for $k = 1,..., K$; and that $\sum_{k=1}^K \lambda_k = 1$. Then*

(1) $x_k \approx y_k$ *for all* $k \Rightarrow \sum_{k=1}^K \lambda_k x_k \approx \sum_{k=1}^K \lambda_k y_k$,

(2) $x_k \succ y_k$ *for some* k *for which* $\lambda_k > 0 \Rightarrow \sum_{k=1}^K \lambda_k x_k \succ \sum_{k=1}^K \lambda_k y_k$.

For the next lemma, $L = \{\lambda x + (1 - \lambda)y : \lambda \in \mathrm{Re}\}$ with $x \neq y$ can be viewed as a line in a finite-dimensional Euclidean space since $x(b)$ and $y(b)$ are nonzero for only a finite number of $b \in B$. If $x', y' \in L$ then the *sense from x' to y' is the same as the sense from x to y* if and only if $\alpha > \beta$ when $x' = \alpha x + (1 - \alpha)y$ and $y' = \beta x + (1 - \beta)y$. As noted later, the conclusion of Lemma 2 does not hold under the weak individual axiom.

LEMMA 2. *Suppose that $\succ$ on X satisfies the moderate individual axiom, and that $x, y \in X$ and $x \neq y$. Let $L = \{\lambda x + (1 - \lambda)y : \lambda \in \mathrm{Re}\}$, and let $L' = L \cap X$. If $x \succ y$ then $x' \succ y'$ whenever $x', y' \in L'$ and the sense from x' to y' is the same as the sense from x to y; if $x \approx y$ then $x' \approx y'$ for all $x', y' \in L'$.*

An appropriate form of the usual expected-utility theorem is given in the next lemma.

LEMMA 3. *Suppose that $\succ$ on X satisfies the strong individual axiom. Then there exists $u: B \to \mathrm{Re}$ such that, for all $x, y \in X$,*

$$x \succ y \Leftrightarrow \sum_B x(b)\, u(b) > \sum_B y(b)\, u(b). \tag{1}$$

Moreover, $v: B \to \mathrm{Re}$ satisfies this in place of u if and only if there are $r, s \in \mathrm{Re}$ with $r > 0$ such that $v(b) = ru(b) + s$ for all $b \in B$.

In contrast to this, the moderate individual axiom does not imply a unidimensional expected-utility result [with $\Rightarrow$ replacing $\Leftrightarrow$ in (1)] since it has no Archimedean condition. For further discussion on this point see Hausner [15], Chipman [5], Aumann [1, 2], Kannai [20] and Fishburn [13; 9, Chap. 9].

4. Admissible Distributions

We presume that there are n voters or individuals in the situation at hand and that $>_i$ is the preference order for individual i. The purpose of this section is to investigate the following dominance notion:

$$x \cdot\!> y \Leftrightarrow x \gtrsim_i y \quad \text{for} \quad i = 1,\ldots, n, \qquad \text{and} \qquad x >_i y \quad \text{for some } i.$$

Recall that $x \gtrsim_i y$ if and only if $x >_i y$ or $x \approx_i y$. Since it seems reasonable that the group would not want to implement a dominated distribution, we shall be concerned with the existence of undominated or admissible distributions.

Definition 4. The distribution $y \in X$ is admissible if and only if $x \cdot\!> y$ for no $x \in X$.

Other definitions of admissibility may suggest themselves. One of these takes y as "admissible" if there is no x for which $x >_i y$ for all i. This is weaker than the given definition and would be handled by methods like those used below. A stronger definition takes y as "admissible" if ($x \succcurlyeq_i y$ for all i, and $x >_i y$ for some i) for no x. This seems too strong for the strict partial orders context since its associated "dominance" relation can be cyclic.

Simple examples show that, even when the strong individual axiom holds for each $>_i$, there may be no admissible distribution when B is infinite. Therefore the rest of this section is mainly concerned with finite B.

In stating our main theorem for admissibility we shall use some subsets of X, two of which are

$$X_0 = \{x \colon x \in X \text{ and } x \text{ is not admissible}\},$$
$$X_1 = \{x \colon x \in X \text{ and } x \text{ is admissible}\}.$$

Clearly, $X = X_0 \cup X_1$ and $X_0 \cap X_1 = \varnothing$. Under the weak individual axiom, these spaces of admissible and inadmissible distributions will be composed of unions of subsets of X which are defined in the following way. Let C be a nonempty subset of B. Then

$$X(C) = \{x\colon x \in X \text{ and } \textstyle\sum_C x(b) = 1\},$$
$$X[C] = \{x\colon x \in X(C) \text{ and } x(b) > 0 \text{ for all } b \in C\}.$$

For a given $C \subseteq B$, both $X(C)$ and $X[C]$ are convex subsets of X: $X(C)$ is the space of all distributions that assign probability 1 to C, and $X[C]$ is the subset of $X(C)$ on which $x(b)$ is positive for *all* $b \in C$. It is natural to refer to $X[C]$ as the interior (or relative interior) of $X(C)$ when C is finite. The boundary (or relative boundary) of $X(C)$ is then $X(C) - X[C]$. If C is a unit subset of B, say $C = \{a\}$, then $X(C) = X[C] = \{a\}$ and the boundary of $\{a\}$ is empty.

A concrete example will help to illustrate these sets. Take $B = \{a, b, c\}$, $n = 2$, and suppose that each of $>_1$ and $>_2$ satisfies the strong individual axiom. We consider two situations, whose utility functions, which satisfy (1) of Lemma 3, are shown in the following matrices. In situation I,

	u_1	u_2
a	3	0
b	1	1
c	0	3

I

	u_1	u_2
a	3	0
b	2	2
c	0	3

II

b is dominated by $\frac{1}{2}a + \frac{1}{2}c$, but b is not dominated in situation II.

To illustrate things graphically, we represent $x \in X$ by the vector $(x(a), x(b), x(c)) \in \mathrm{Re}^3$, so that X can be viewed as the planar simplex shown at the top of Fig. 1. The admissible distributions in X_1 for each of

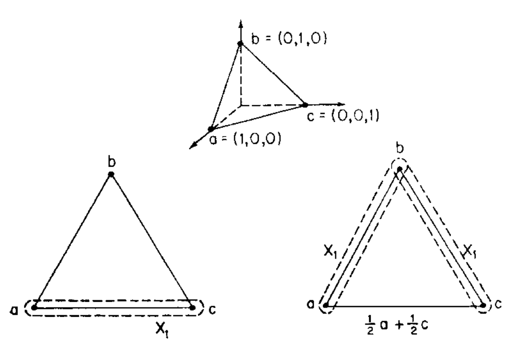

FIG. 1 Admissible distributions with three basic alternatives.

situations I and II are enclosed by the dashed lines. For situation I, $X_1 = X(\{a, c\})$: that is, $x \in X$ is admissible if and only if $x(a) + x(c) = 1$, or $x(b) = 0$, as is easily shown from the u_i values. In addition $X_0 = X - X_1 = X[\{a, b, c\}] \cup X[\{a, b\}] \cup X[\{b, c\}] \cup X[\{b\}]$.

In general, a subset Y of X is *convex* if and only if $x, y \in Y$ and $\lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda) y \in Y$. Both X_0 and X_1 are convex in situation I. However, only X_0 is convex in situation II where $X_0 = X[\{a, b, c\}] \cup X[\{a, c\}]$ and $X_1 = X(\{a, b\}) \cup X(\{b, c\})$.

We now state our main theorem for admissibility.

THEOREM 1. *Suppose that B is nonempty and finite and that each $>_i$ satisfies the weak individual axiom. Then there is a nonempty set $\mathscr{B}_1$ of nonempty subsets of B such that*

(1) $C, C' \in \mathscr{B}_1$ *and* $C \neq C' \Rightarrow$ *not* $(C \subseteq C')$ *and not* $(C' \subseteq C)$,

(2) $X_1 = \bigcup\{X(C)\colon C \in \mathscr{B}_1\}$.

Moreover, if $X_0 \neq \varnothing$ then there is a nonempty set $\mathscr{B}_0$ of nonempty subsets of B such that

(3) $C \in \mathscr{B}_0$ *and* $C \subseteq C' \subseteq B \Rightarrow C' \in \mathscr{B}_0$,

(4) $C_0 \in \mathscr{B}_0$ *and* $C_1 \in \mathscr{B}_1 \Rightarrow$ *not* $(C_0 \subseteq C_1)$,

(5) X_0 *is convex, and* $X_0 = \bigcup\{X[C]\colon C \in \mathscr{B}_0\}$.

The following aspects of the theorem should be noted. First, (2) and the structure of $\mathscr{B}_1$ say that $X_1 \neq \varnothing$, and $b \in X_1$ for some $b \in B$. Since $X(C) \subseteq X(C')$ if $C \subseteq C'$, (1) restricts $\mathscr{B}_1$ to the minimal set of subsets of the form $X(C)$ whose union equals X_1. We know already that X_1 may not be convex. It will be shown later that there may be $C, C' \in \mathscr{B}_1$ for which $C \cap C' = \varnothing$.

Second, if X_0 is not empty then it is convex and is equal to the union of the interiors of the $X(C)$ for $C \in \mathscr{B}_0$. If $C \neq C'$ then $X[C] \cap X[C'] = \varnothing$. Parts (3) and (5) say that if x is inadmissible and if $y(b) > 0$ for every b for which $x(b) > 0$ then y is inadmissible. Part (4) simply reflects the fact that $X_0 \cap X_1 = \varnothing$.

Proof of Theorem 1. The hypotheses of the theorem are assumed to hold. We begin by noting that $b \in X_1$ for some $b \in B$. This is stated in Lemma 6 below, whose proof will be based on Lemmas 4 and 5. Lemma 4, which in the theory of Markov processes guarantees the existence of a stationary distribution p in the finite context, will not be proved here: one proof is given by Rosenblatt [24, pp. 44–52]; other proofs can be based on a theorem of the alternative [9] or on Kakutani's fixed-point theorem [19].

LEMMA 4. *Suppose that* $x_1, \ldots, x_m$ *are probability distributions on* $\{1, \ldots, m\}$. *Then there is a probability distribution* p *on* $\{1, \ldots, m\}$ *such that*

$$p(k) = \sum_{j=1}^{m} p(j)\, x_j(k) \qquad \textit{for} \quad k = 1, \ldots, m.$$

LEMMA 5. *Suppose that* $x_k, y_k \in X$ *and* $x_k \cdot\!> y_k$ *and* $\lambda_k \geqslant 0$ *for* $k = 1, \ldots, m$, *with* $\sum_{k=1}^{m} \lambda_k = 1$. *Then* $\sum_{k=1}^{m} \lambda_k x_k \cdot\!> \sum_{k=1}^{m} \lambda_k y_k$.

Proof. Since $x_k \cdot\!> y_k$ for all k, $x_k \gtrsim_i y_k$ for all i and k, and therefore $\sum \lambda_k x_k \gtrsim_i \sum \lambda_k y_k$ for all i by Lemma 1. For some $\lambda_k > 0$ there is an i such that $x_k \succ_i y_k$. Hence $\sum \lambda_k x_k \succ_i \sum \lambda_k y_k$ for this i, by Lemma 1(2). Hence $\sum \lambda_k x_k \cdot\!> \sum \lambda_k y_k$ by the definition of $\cdot\!>$.

LEMMA 6. *There exists a* $b \in B$ *such that* b *is admissible.*

Proof. Let $B = \{b_1, \ldots, b_m\}$. Contrary to Lemma 6, suppose that every b_j is dominated, with $x_j \in X$ and $x_j \cdot\!> b_j$ for $j = 1, \ldots, m$. Let $p \in X$ be as guaranteed by Lemma 4:

$$p(b_k) = \sum_{j=1}^{m} p(b_j)\, x_j(b_k) \qquad \text{for} \quad k = 1, \ldots, m.$$

Then $p = \sum p(b_j)\, b_j = \sum p(b_j)\, x_j$, so that $p \cdot\!> p$ by Lemma 5. But this contradicts the irreflexivity of some $\succ_i$, and hence it is false that every b_j is dominated.

If $X_0 = \varnothing$ then $X_1 = X$ and the theorem holds with $\mathscr{B}_1 = \{B\}$. We now consider the structure of X_0 when it is not empty.

LEMMA 7. $x \in X_0$ *and* $\lambda \in (0, 1) \Rightarrow \lambda x + (1 - \lambda) y \in X_0$.

Proof. Suppose that $z \cdot\!> x$ and $0 < \lambda < 1$. Then, by the proof method for Lemma 6, $\lambda z + (1 - \lambda) y \cdot\!> \lambda x + (1 - \lambda) y$.

Suppose that $X_0 \neq \varnothing$. Then Lemma 7 says that X_0 is convex, and that

$$x \in X_0 \qquad \text{and} \qquad x, y \in X[C] \Rightarrow y \in X_0,$$

since any $y \neq x$ that is also in the interior of $X(C)$ equals a nontrivial combination of x and a point on the boundary of $X(C)$. Moreover, if $x \in X_0$, $x \in X[C]$ and $C \subset C'$, then some $y \in X[C']$ must be in X_0, and hence every $y \in X[C']$ is in X_0. This establishes (3) and (5) of the theorem. Parts (1), (2) and (4) then follow readily from the fact that $X_1 = X - X_0$.

Dominance by Admissible Distributions

Under the weak individual axiom, it can be false that each $y \in X_0$ is dominated by some $x \in X_1$. For example, let $B = \{a, b\}$ and let $x \in [0, 1]$ represent the distribution that has probability x for a and probability $1 - x$ for b. Taking $n = 1$, let $\succ = \succ_1$ be as follows: $x \succ y \succ 1$ whenever $0 < x < y < 1$, and $0 \sim x$ for all $x \in (0, 1]$. This is easily seen to satisfy the weak individual axiom but not the conclusion of Lemma 2, and since 0 is the only undominated point, $X_1 = \{b\}$. But $b \cdot\!> x$ for no $x \in (0, 1]$, and no point in X_0 is dominated by a point in X_1.

This anomaly is removed when the moderate individual axiom is used.

THEOREM 2. *Suppose that B is nonempty and finite, and that each $\succ_i$ satisfies the moderate individual axiom. Then $y \in X_0 \Rightarrow x \cdot\!> y$ for some $x \in X_1$.*

Proof. Let the hypotheses hold and, contrary to the conclusion, suppose that $y_1 \in X_0$ and no admissible distribution dominates y_1. Beginning with y_1, we construct a sequence $y_1, y_2, y_3, \ldots$ Notationally, we shall let $C_k \subseteq B$ be such that $y_k \in X[C_k]$.

Given y_1, determine y_2 as follows. If there is an $x \in X[C_1]$ such that $x \cdot\!> y_1$, take y_2 as the point on the line $\{\lambda x + (1 - \lambda) y_1 : \lambda \in \mathrm{Re}\}$ that is farthest from y_1 on the x side of y_1 and is still in X. Then, by Lemma 2, $y_2 \cdot\!> y_1$ and, by construction with $y_2 \in X[C_2]$, $C_2 \subset C_1$. [That is, y_2 is on the boundary of $X(C_1)$ and hence will be in the interior of $X(C_2)$ for some $C_2 \subset C_1$.] On the other hand, if $x \cdot\!> y_1$ for no $x \in X[C_1]$, let y_2 be any point in X_0 that dominates y_1. Since X_0 is convex, C_2 will not be a subset of C_1 in this case.

Given y_2, the same procedure (with y_2 replacing y_1) is used to obtain $y_3 \cdot\!> y_2$. The construction proceeds in the same way for each y_k. Since $\cdot\!>$ is transitive (why?), $y_k \cdot\!> y_j$ whenever $k > j$. Moreover, since $\cdot\!>$ is irreflexive, $y_k \neq y_j$ whenever $k \neq j$.

Suppose we are at y_j. Since B is finite, and since $C_{k+1} \subset C_k$ if $x \cdot\!> y_k$ for some $x \in X[C_k]$, after a finite number of steps we must reach a $k > j$ such that $x \cdot\!> y_k$ for no $x \in X[C_k]$. And since $y_m \cdot\!> y_k$ for $m > k$, this implies that $y_m \notin X[C_k]$ *for all* $m > k$. It follows that there is an infinite sequence $k_1, k_2, \ldots$ with $k_1 < k_2 < \cdots$ such that, for each $r \in \{1, 2, \ldots\}$, $y_m \notin X[C_{k_r}]$ for all $m > k_r$. But this implies that no two C_{k_r} are identical and hence that their number is infinite. Since B has only a finite number of subsets, we have a contradiction. Hence $y_1 \in X_0 \Rightarrow x \cdot\!> y_1$ for some $x \in X_1$.

A Conjecture and Counterexample

As shown on Fig. 1, X_1 need not be convex. However, the first part of Theorem 1 states that X_1 is the union of several maximal convex sets, namely, the $X(C)$ for $C \in \mathscr{B}_1$. It might therefore be conjectured that each two subsets of B that are in $\mathscr{B}_1$ contain a common basic alternative, or $C \cap C' \neq \varnothing$ when $C, C' \in \mathscr{B}_1$.

Simple examples show that this can be false when only the weak individual axiom is used and B has more than one element. Moreover, as we shall now show, it can be false even when the strong individual axiom is used, provided that B has more than three elements.

Let $B = \{a, b, c, d\}$, $n = 2$ and suppose the following $u_i : B \to \mathrm{Re}$ satisfy (1) of Lemma 3 for $i = 1, 2$. Then, as you can easily verify, each of a, b, c and d

	u_1	u_2
a	5	0
b	0	5
c	3	3
d	4	2

is in X_1. Moreover, $c \cdot\!> \frac{1}{2}a + \frac{1}{2}b$, $d \cdot\!> \frac{1}{2}a + \frac{1}{2}c$ and $c \cdot\!> \frac{1}{4}b + \frac{3}{4}d$. By Theorem 1, this yields $X_1 \subseteq X(\{a, d\}) \cup X(\{c, d\}) \cup X(\{b, c\})$. In fact, X_1

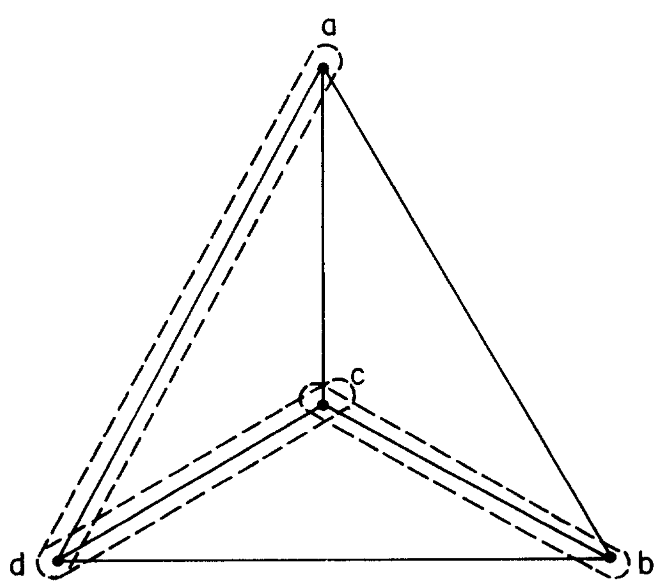

FIG. 2 Admissible distributions with four basic alternatives.

can be shown to equal this union, so that $\mathscr{B}_1 = \{\{a, d\}, \{c, d\}, \{b, c\}\}$ with $\{a, d\} \cap \{b, c\} = \varnothing$.

This result is shown on Fig. 2. Since X is effectively 3-dimensional, it can be viewed as a regular tetrahedron with c lying above the plane that contains a, b and d. As before, the points in X_1, along three edges of the tetrahedron, are enclosed by dashed lines.

5. The Choice Set

Within the context of this paper, the usual notion of a choice set applies. This is to be a nonempty subset of X whose elements are, roughly speaking, socially satisfactory. If X^* is the choice set from X then some $x \in X^*$ will be implemented.

As suggested in the preceding section, one reasonable requirement for the choice set X^* is that $X^* \subseteq X_1$, provided that $X_1 \neq \varnothing$. That is, each element in the choice set should be admissible.

Another condition for X^* is that it be convex. Although this does not seem as compelling as admissibility, it has some appeal. Suppose for example that $x, y \in X^*$ and $x \neq y$. Then the distributions x and y both appear "fair" for implementation despite the fact that some individuals may prefer x to y while others prefer y to x. Hence $\lambda x + (1 - \lambda) y$ with $0 < \lambda < 1$ might also seem "fair", since when viewed as a two-stage lottery it results in either x or y at the first stage according to the probabilities λ and $1 - \lambda$.

If, under the hypotheses of Theorem 1, both admissibility and convexity are imposed on X^* then X^* will have to be a proper subset of X_1 if X_1 is not convex.

6. Simple Majorities

Naturally, conditions other than admissibility and convexity will usually play a role in determining the choice set X^*. For example, let xPy mean that more individuals prefer x to y than prefer y to x. Then if there is an $x \in X$ such that xPy for all $y \neq x$ in X, it might seem reasonable to take $X^* = \{x\}$. Or if there is a $b \in B$ such that bPa for all $a \neq b$ in B, and if b is admissible, then it might be suggested as the social choice.

We shall conclude this study with three theorems that involve the strict simple majority relation P in the lottery context. The weak individual axiom is used in the first two, and a somewhat stronger axiom is used in the third. The first theorem shows that if xPy for all $y \neq x$ in X, then x amounts to a basic alternative. The latter two are concerned with single-peaked preferences. It is not assumed that B is finite.

THEOREM 3. *Suppose that each $\succ_i$ on X satisfies the weak individual axiom and that $x \in X$ is such that xPy for every $y \in X - \{x\}$. Then $x = b$ for some $b \in B$.*

Proof. Let the hypotheses hold, and take $C = \{b: b \in B \text{ and } x(b) > 0\}$ so that C is finite with $x \in X[C]$. We are to show that C is a unit subset of B.

To the contrary, suppose that C has more than one element. Because x is in the interior of $X(C)$, we can choose $y \in X[C]$ sufficiently close to but different than x so that there are $t, v \in X[C]$ such that

$$x = \lambda t + (1 - \lambda) y$$
$$y = \lambda v + (1 - \lambda) x$$

with $\lambda \in (0, 1)$ and small enough that the results to be described will hold. $\lambda = 0.2$ is a satisfactory value. This is shown on Fig. 3, where $r_1 = \lambda t + (1 - \lambda)x$. An appropriate $\alpha \in (0, 1)$ will give $\alpha r_1 + (1 - \alpha)v = x$, and for this α we take $r_2 = \alpha x + (1 - \alpha)v$. As shown in the figure, r_2 is between x and y. Then $r_3 = \alpha r_2 + (1 - \alpha)v$, with r_3 to the right of y.

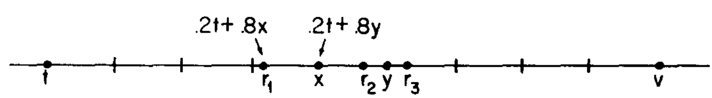

FIG. 3 Construction for proof of Theorem 3.

Suppose that $x \succ_i y$. Then, by the weak individual axiom, $r_1 = \lambda t + (1 - \lambda)x \succ_i \lambda t + (1 - \lambda) y = x$, so that $r_1 \succ_i x$. Conversely, suppose that $r_1 \succ_i x$. Then with the combinations described above, the weak individual axiom implies $x \succ_i r_2$, then $r_2 \succ_i r_3$, so that $x \succ_i r_3$ by transitivity. Since y is a convex combination of x and r_3, the weak individual axiom gives $x \succ_i y$. Hence $x \succ_i y \Leftrightarrow r_1 \succ_i x$. Reversing $\succ_i$ in each step here gives $y \succ_i x \Leftrightarrow x \succ_i r_1$. Therefore $xPy \Leftrightarrow r_1Px$. But this contradicts the hypothesis that xPy for all $y \in X - \{x\}$. Therefore C has only one element.

Single-Peaked Preferences

We shall use the following generalized definition for single-peaked preferences on the basic alternatives in B. A binary relation $<$ on B is a *linear order* if and only if it is asymmetric, transitive and weakly connected ($a \neq b \Rightarrow a < b$ or $b < a$).

DEFINITION 5. The individual preference orders $\succ_1, \ldots, \succ_n$ on X are single-peaked for B if and only if there is a linear order $<$ on B such that, for each $i \in \{1, \ldots, n\}$, there are disjoint subsets A_i, B_i and C_i (one or two of which can be empty) of B such that

(1) $A_i \cup B_i \cup C_i = B$

(2) $(a, b) \in A_i \times B_i \cup B_i \times C_i \cup A_i \times C_i \Rightarrow a < b$

(3a) $a, b \in A_i$ and $a < b \Rightarrow b \succ_i a$

(3b) $a, b \in B_i \Rightarrow a \sim_i b$

(3c) $a, b \in C_i$ and $b < a \Rightarrow b \succ_i a$

(4) $a, b \in (A_i \cup C_i) \times B_i \Rightarrow b \succcurlyeq_i a$

(5) $a < b < c$ and $a \sim_i b$ and $b \sim_i c \Rightarrow a \sim_i c$.

This definition is designed for the context where each $\succ_i$ is a strict partial order, with B finite or infinite. Using proof methods like those in Fishburn [10] it is not hard to show that when individual preferences are single-peaked for B and each $\succ_i$ is a strict partial order then P is transitive on B. Moreover, if n is odd and the restriction of each $\succ_i$ on B (i.e., on the distributions with probability 1 for a basic alternative) is linear, then the restriction of P on B is linear. In this case some basic alternative must have a strict simple majority over every other basic alternative, provided that B is finite.

Our final theorems are concerned with the admissibility of a basic alternative that has a strict simple majority over every other basic alternative. As seen by the $\{a, b, c, d\}$ example of Section 1, where dPa, dPb and dPc, such an alternative may be inadmissible. However, it is easily checked that individual preferences are not single-peaked for $\{a, b, c, d\}$ in that example. This contrasts with the following results.

THEOREM 4. *Suppose that each $\succ_i$ on X satisfies the weak individual axiom and that individual preferences are single-peaked for B under the linear order $<$ on B. Suppose further that for each i there is an $a_i \in B$ such that $a_i \succ_i b$ for all $b \in B - \{a_i\}$. Then, if there is a $b \in B$ such that bPa for all $a \neq b$ in B, b is admissible.*

THEOREM 5. *Suppose that each $\succ_i$ on X is a weak order that satisfies the moderate individual axiom and that individual preferences are single-peaked for B under the linear order $<$ on B. Then, if there is a $b \in B$ such that bPa for all $a \neq b$ in B, b is admissible.*

Proof of Theorem 4. Let the hypotheses hold and assume that bPa for all $a \in B - \{b\}$. Suppose first that $b = a_i$ for some i so that $b \succ_i a$ for all $a \in B - \{b\}$. Let $x \in X$ with $x \neq b$. If $x(b) = 0$ then, by Lemma 1, $\sum\{x(a)b: a \in B\} >_i \sum\{x(a)a: a \in B\}$, or $b >_i x$. If $0 < x(b) < 1$ then $x = x(b)b + [1 - x(b)]\, x'$ with $x'(b) = 0$. As just shown, $b >_i x'$. Therefore $b = [1 - x(b)]b + x(b)b >_i [1 - x(b)]\, x' + x(b)b$, or $b >_i x$. Hence $b >_i x$ for all $x \neq b$, so that b cannot be dominated.

Suppose next that $b = a_i$ for no i. Let $a, c \in \{a_1, \ldots, a_n\}$ satisfy [$a < b$ and $a < a_i < b$ for no a_i] and [$b < c$ and $b < a_i < c$ for no a_i]. Since bPa_i for all i, both a and c exist. Let m and $n - m$ be, respectively, the number of i for which $a_i < b$ and for which $b < a_i$. Then $bPa \Rightarrow m > n - m$, and $bPc \Rightarrow n - m > m$, a contradiction. Hence the case supposed in this paragraph cannot arise.

Proof of Theorem 5. Let the hypotheses hold and assume that bPa for all $a \in B - \{b\}$. We shall let

$$I = \{i: b \succcurlyeq_i a \text{ for all } a \in B - \{b\}\},$$
$$B_i = \{a: a \in B \text{ and } a \sim_i b\} \qquad \text{for each} \quad i \in I.$$

Suppose first that $I = \varnothing$. Then either b is an extreme point of the linearly ordered set $(B, <)$, in which case $a \succcurlyeq_i b$ for all a and i, thus contradicting bPa, or else $a < b < c$ for some $a, c \in B$. In this case, with $I = \varnothing$, let m individuals have points $< b$ that are preferred to b, and let $n - m$ individuals have points $> b$ that are preferred to b. Then it is easily seen that there exist $a, c \in B$ such that $a < b < c$ with $a >_i b >_i c$ for m individuals and $c >_i b >_i a$ for $n - m$. But this contradicts (bPa and bPc). Therefore $I = \varnothing$ must be false.

Given $I \neq \varnothing$, suppose that, contrary to the theorem, $x \cdot\!> b$ for some $x \in X$. Since this requires $x \succcurlyeq_i b$ for each i, it follows that $x(B_i) = \sum\{x(a): a \in B_i\} = 1$ for each $i \in I$. Therefore, with $C = \bigcap\{B_i : i \in I\}$, $x(C) = 1$. Without loss in generality we can take $x(b) = 0$. Then suppose first that $c < b$ for all $c \in C$, or else $b < c$ for all $c \in C$. To have bPc for each $c \in C$ in the latter case, there must be an i such that $b \in B_i \cup C_i$ (see Definition 5) with $C \subseteq C_i$. But then $b >_i c$ for all $c \in C$ and hence $b >_i x$ by $x(C) = 1$ and Lemma 1, thus contradicting $x \cdot\!> b$. Suppose next that $a < b < c$ for some $a, c \in C$. Then there exist such a and c for which $a \sim_i b \sim_i c$ for all $i \in I$, and $a >_i b >_i c$ or $c >_i b >_i a$ for each $i \notin I$. But this contradicts (bPc and bPa), and hence it is false that $x \cdot\!> b$ for some $x \in X$.

Appendix

Proof of Lemma 1. Suppose first that $x_k \approx y_k$ for $k = 1, \ldots, K$. We use induction on K. For $K = 2$, part 2 of the weak individual axiom (see

Definition 1) gives $\lambda_1 x_1 + \lambda_2 x_2 \approx \lambda_1 y_1 + \lambda_2 x_2 \approx \lambda_1 y_1 + \lambda_2 y_2$. Since $\approx$ is transitive when $\succ$ is a strict partial order, $\lambda_1 x_1 + \lambda_2 x_2 \approx \lambda_1 y_1 + \lambda_2 y_2$. (If $\lambda_1 = 1$ or $\lambda_2 = 1$, this conclusion is immediate.) Now suppose that (1) of Lemma 1 is true for $K = 2,..., m - 1$. For the case of $K = m$ take $0 < \lambda_m < 1$, by resubscripting if necessary. Then by the induction hypothesis,

$$(1 - \lambda_m)^{-1} \sum_{k=1}^{m-1} \lambda_k x_k \approx (1 - \lambda_m)^{-1} \sum_{k=1}^{m-1} \lambda_k y_k,$$

$$(1 - \lambda_m)\left[(1 - \lambda_m)^{-1} \sum_{k=1}^{m-1} \lambda_k x_k\right] + \lambda_m x_m \approx (1 - \lambda_m)\left[(1 - \lambda_m)^{-1} \sum_{k=1}^{m-1} \lambda_k y_k\right] + \lambda_m y_m,$$

the latter of which is the same as $\sum_1^m \lambda_k x_k \approx \sum_1^m \lambda_k y_k$.

For part (2) of Lemma 1, assume for definiteness that $x_K \succ y_K$ and $0 < \lambda_K < 1$. (If $\lambda_K = 1$, the conclusion is obvious.) Then, with $K = 2$, the weak individual axiom gives $\lambda_1 x_1 + \lambda_2 x_2 \succ \lambda_1 y_1 + \lambda_2 y_2$. Proceeding by induction as before, $\sum_1^m \lambda_k x_k \succ \sum_1^m \lambda_k y_k$ follows in the obvious manner when $m = K \geqslant 2$.

Proof of Lemma 2. Given $x \neq y$, $L' = L \cap X$ is the segment of $L = \{\lambda x + (1 - \lambda) y : \lambda \in \mathrm{Re}\}$ in X. Let x^* and y^* be the extreme points in L' so that $L' = \{\lambda x^* + (1 - \lambda) y^* : 0 \leqslant \lambda \leqslant 1\}$, with $\alpha > \beta$ when $x = \alpha x^* + (1 - \alpha) y^*$ and $y = \beta x^* + (1 - \beta) y^*$.

Suppose that $x \succ y$. With $\alpha > \beta$, $x \succ y$ is the same as

$$\beta x^* + (1 - \beta)\left[\frac{\alpha - \beta}{1 - \beta} x^* + \frac{1 - \alpha}{1 - \beta} y^*\right] \succ \beta x^* + (1 - \beta) y^*,$$

so that $\gamma x^* + (1 - \gamma) y^* \succ y^*$ by the moderate individual axiom when $\gamma = (\alpha - \beta)/(1 - \beta)$. Another application of the moderate individual axiom then gives $x^* \succ y^*$.

Given $x^* \succ y^*$, let $x' = px^* + (1 - p) y^*$ and $y' = qx^* + (1 - q) y^*$ with $1 \geqslant p > q \geqslant 0$. Then $px^* + (1 - p) y^* \succ y^*$ by the weak individual axiom, and if $q > 0$ then

$$\begin{aligned}(q/p)[px^* + (1 - p) y^*] + (1 - q/p)[px^* + (1 - p) y^*] \\ \succ (q/p)[px^* + (1 - p) y^*] + (1 - q/p) y^*,\end{aligned}$$

or $px^* + (1 - p) y^* \succ qx^* + (1 - q) y^*$.

The proof for $\approx$ is similar.

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