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Author(s): Kenneth A. Shepsle

Source: *The Quarterly Journal of Economics*, Nov., 1970, Vol. 84, No. 4 (Nov., 1970), pp. 705-709

Published by: Oxford University Press

Stable URL: https://www.jstor.org/stable/1880854

A NOTE ON ZECKHAUSER'S "MAJORITY RULE WITH LOTTERIES ON ALTERNATIVES": THE CASE OF THE PARADOX OF VOTING *

KENNETH A. SHEPSLE

I. INTRODUCTORY REMARKS

Recently, Professor Richard Zeckhauser traced some of the fascinating voting implications of a majority-rule collectivity that permits lotteries over basic social alternatives.[1] In this note we reformulate the problem in order to specify some additional implications.[2]

Consider a set of three voters, $V = \{v_1, v_2, v_3\}$, and a set of three alternatives, $A = \{a_1, a_2, a_3\}$, one of which is chosen by the collectivity V by majority rule. Each $v_i \epsilon V$ is assumed to have a connected, transitive preference ordering over the $a_j \epsilon A$. Duncan Black has shown that the condition of single peakedness is sufficient for the existence of a majority alternative, where a *majority alternative* is an element $a_i \epsilon$A preferred by some majority to every other $a_j \epsilon A$.[3] The majority alternative — call it a_{med} — is the median most preferred alternative.

Zeckhauser has noted that under certain conditions a lottery over some subset (not necessarily proper) of the alternatives can defeat a_{med}. He further notes that if all voters have single-peaked, strong preferences, then one of the component sure prospects of this lottery can, itself, defeat the lottery. Thus, it follows that if lotteries over basic social alternatives are admissible, then a majority

* I would like to thank Professors Richard G. Niemi and William H. Riker, as well as the anonymous referee, for their helpful comments.

1. Richard Zeckhauser, "Majority Rule with Lotteries on Alternatives," this *Journal*, Vol. 83 (Nov. 1969), pp. 696–703.

2. This entire argument is developed at length in Kenneth A. Shepsle, "Essays on Risky Choice in Electoral Competition," Ph.D. thesis, University of Rochester, 1970.

3. Duncan Black, *The Theory of Committees and Elections* (Cambridge: Cambridge University Press, 1963), pp. 14–25. This result generalizes to any number of alternatives and any odd number of voters.

alternative [4] does not exist in the absence of additional restrictions; i.e., majority rule leads to intransitive or cyclical choice.[5]

In this paper we examine the case in which the basic social alternatives cycle. We then trace the implications of majority rule when lotteries are admitted. This case, one which Zeckhauser did not consider, suggests a different role for uncertainty and intensity of preference than was found in the single-peakedness case.

II. The Paradox of Voting Under Uncertainty: A Special Case

Let the set $V=\{v_1, v_2, v_3\}$ be an electorate to which two candidates, X and Y, appeal for votes. Whichever candidate receives a majority of the votes wins the election. The election hinges on one issue, A, for which there are three possible positions, i.e., $A=\{a_1, a_2, a_3\}$. Each voter has a preference ordering on A and votes for that candidate, Mr. X or Mr. Y, who advocates the more preferred position. Suppose that the set of preference orderings of the voters is one that generates a cyclical majority.

v_1	v_2	v_3
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

In this case no majority alternative exists. However, a lottery may exist that defeats *all* of the certain alternatives. If one of the candidates is restricted to the set of certain alternatives (degenerate lotteries) and the other candidate, who is not so restricted, does not know which alternative the first candidate will choose, then it is reasonable for him to examine whether a lottery strategy will win the election, *regardless of his opponent's choice.*

In order to show our result, we convert the preference orderings of the v_i to von Neumann–Morgenstern utility schedules. This is done in Table I. We suggest that a lottery (p_1, p_2, p_3) over A defeats all of the certain alternatives if and only if the expected utility of the lottery exceeds the utility of the second-ranked alternative for each v_i.[6] That is, (p_1, p_2, p_3) defeats each $a_i \epsilon A$ if and only if

(1) $p_1(1)+p_2(k)+p_3(0)>k$

(2) $p_1(0)+p_2(1)+p_3(m)>m$

(3) $p_1(n)+p_2(0)+p_3(1)>n.$

4. A "majority alternative" in the case of choice among lotteries over basic social alternatives is a lottery — degenerate or nondegenerate — preferred by some majority to all other admissible lotteries.

5. Zeckhauser, *op. cit.,* p. 698.

6. This result is proved in Shepsle, *op. cit.,* Ch. 5.

TABLE I

VOTERS' UTILITY SCHEDULES

		V			
		v_1	v_2	v_3	
	a_1	1	0	n	
					$0 < k < 1$
A	a_2	k	1	0	$0 < m < 1$
					$0 < n < 1$
	a_3	0	m	1	

To see that this is true, suppose that the "certain" candidate (Mr. X) advocates a_1 and the "risky" candidate (Mr. Y) advocates the lottery (p_1, p_2, p_3). From Table I it is seen that v_1 clearly prefers X and v_2 clearly prefers Y. However, from (3) it follows that v_3 prefers Y. Mr. Y wins. If X advocates a_2, then v_2 prefers X and v_3 prefers Y, while from (1) v_1 prefers Y. Mr. Y wins again. Finally, if X advocates a_3 he receives v_3's vote, loses v_1's vote, and from (2) loses v_2's vote. That is, Mr. Y and his lottery win, regardless of what Mr. X does.

Given this result we now ask whether such a lottery exists. A lottery exists if (1)–(3) are simultaneously satisfied, subject to the axioms of probability.[7] A simultaneous solution of inequalities (1)–(3) implies the following necessary and sufficient condition for existence:

(4) if $kmn \leqslant (1-k)(1-m)(1-n)$, then a lottery (p_1, p_2, p_3), consistent with (1)–(3), exists.

Upper limits for k for various values of m and n are given in Table II. A quick glance at this table reveals an interesting fact that may not be obvious from the inequality in (4). At least one of the elements in the triple (k, m, n) must not exceed 0.5. Furthermore, there is a monotonically decreasing relationship between any two of the elements when the third is held constant. If we interpret a utility value of 0.5 or smaller for a voter's second-ranked alternative as an expression of intense preference, then the inequality in (4) suggests that in the case of cyclical majorities with intense preferences, there are incentives for political candidates to act ambiguously, *viz.*, to offer lotteries to their constituents.

7. In addition, the result must be invariant under positive linear transformations of utility.

TABLE II

UPPER LIMITS ON k SUCH THAT A DOMINANT LOTTERY EXISTS

		n								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0.1	.988	.974	.954	.931	.900	.858	.794	.693	.500
	.2	—	.941	.904	.858	.800	.727	.632	.500	.308
	.3		—	.845	.778	.700	.609	.500	.389	.206
	.4			—	.693	.600	.500	.392	.272	.143
m	.5				—	.500	.400	.300	.200	.100
	.6					—	.308	.222	.143	.069
	.7						—	.155	.097	.045
	.8							—	.059	.027
	.9								—	.012

III. THE UNCONSTRAINED CASE AND THE IMPORTANCE OF STRATEGY CONSTRAINTS

In the last several paragraphs we examined a rather special case, namely, that in which one of the candidates is constrained to a particular subset of feasible electoral strategies — the set of degenerate lotteries. In the unconstrained case a not too surprising result obtains. The set of nondegenerate lotteries cycle, even if inequality (4) is satisfied. A proof employing the "objection" concept of Aumann and Maschler is easily constructed. It shows that, for any feasible lottery, an objection may be constructed by a simple reallocation of probability.[8]

The juxtaposition of special case and general unconstrained case is instructive. It suggests that constraints may eliminate the devastating effects of the cyclical majority problem by admitting strategies in one candidate's constrained set that dominate all strategies in the other candidate's set. Since real world political candidates are often constrained by the historical record of the parties that they represent, by the preferences of important supporters, e.g., financial contributors, by personal histories, and by their electoral role (incumbent or challenger), it appears that the concept of strategic constraint recommends serious consideration in models of political competition. More generally, the concept focuses at-

8. See Shepsle, *op. cit.*, for the proof.

tention on the importance of "rules of entry" — rules that determine the composition of the choice set A.

A number of interesting considerations are raised by Zeckhauser's initial probes. Writing in the welfare economics tradition of K. J. Arrow and Black, he isolates additional features of the preference structure needed to avoid cyclical majorities involving lotteries. We are interested in a related question, but view it from a different perspective. We might call it the "constrained social welfare problem." In the rhetoric of welfare economics, this perspective involves a weakening of the connectivity axiom. That is, *political* constraints do not permit all possible dyads of alternatives to be processed by the preference aggregation principle. Political parties and candidates are restricted in the alternatives they offer the electorate.

We are presently engaged in an examination of the implications of systematic, well-specified constraints for the majority rule problem under uncertainty. This permits us to retrace the welfare economist's footsteps in order to put the "political" back into political economy!

Washington University, St. Louis