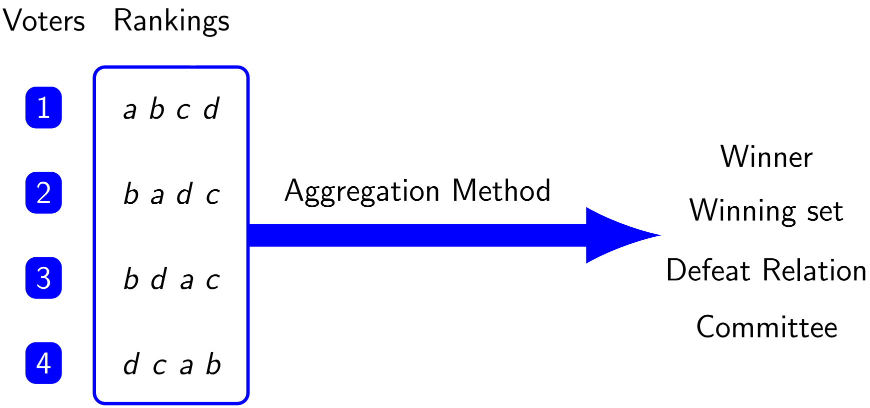
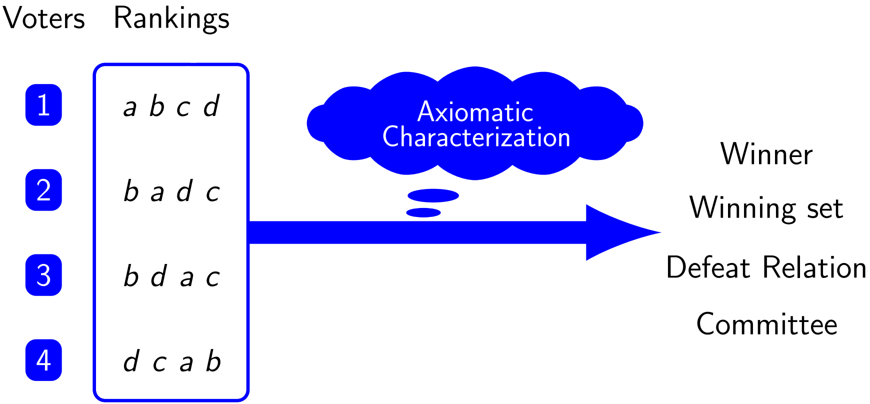
New Perspectives on Social Choice

Eric Pacuit, University of Maryland

Lecture 5: Probabilistic Social Choice

PHIL 808K


Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee


Voters
Rankings
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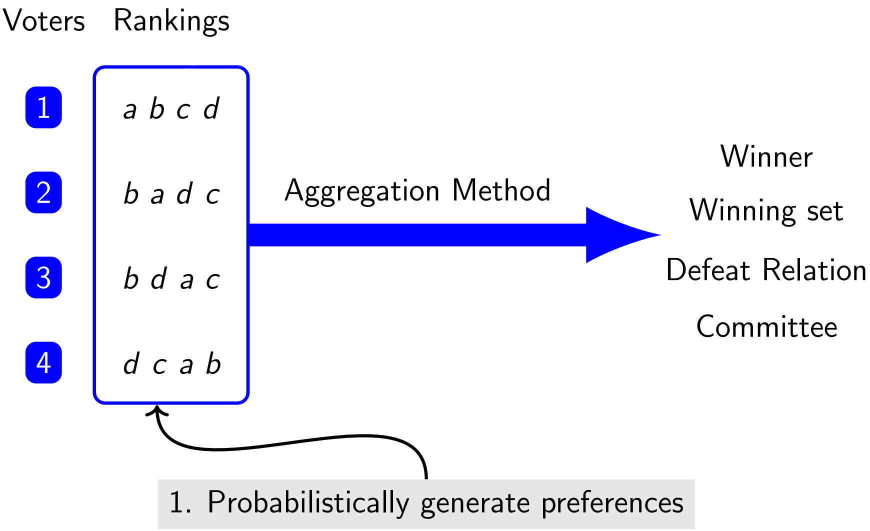
$b \; a \; d \; c$
$b \; d \; a \; c$
$d \; c \; a \; b$
Axiomatic Characterization
Winner
Winning set
Defeat Relation
Committee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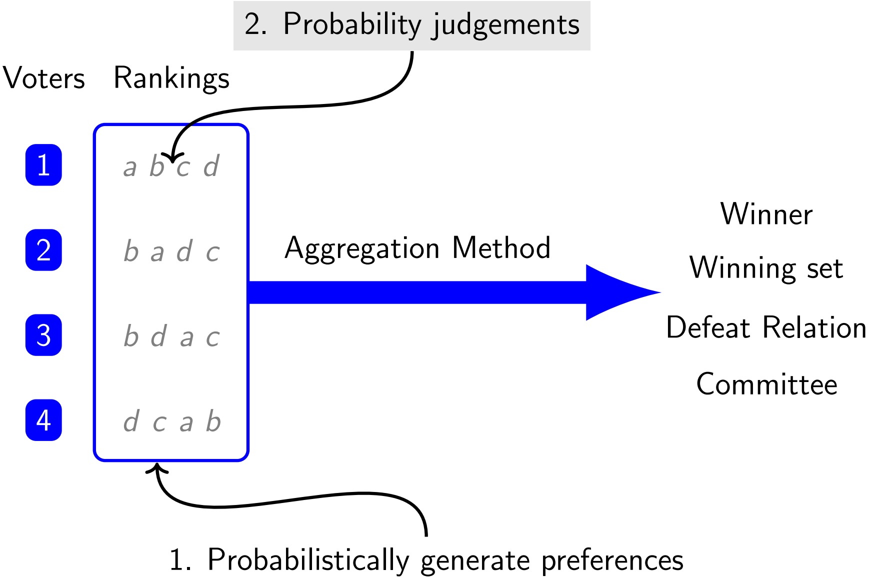
Committee
1. Probabilistically generate preferences


2. Probability judgements
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a b c d
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b a d c
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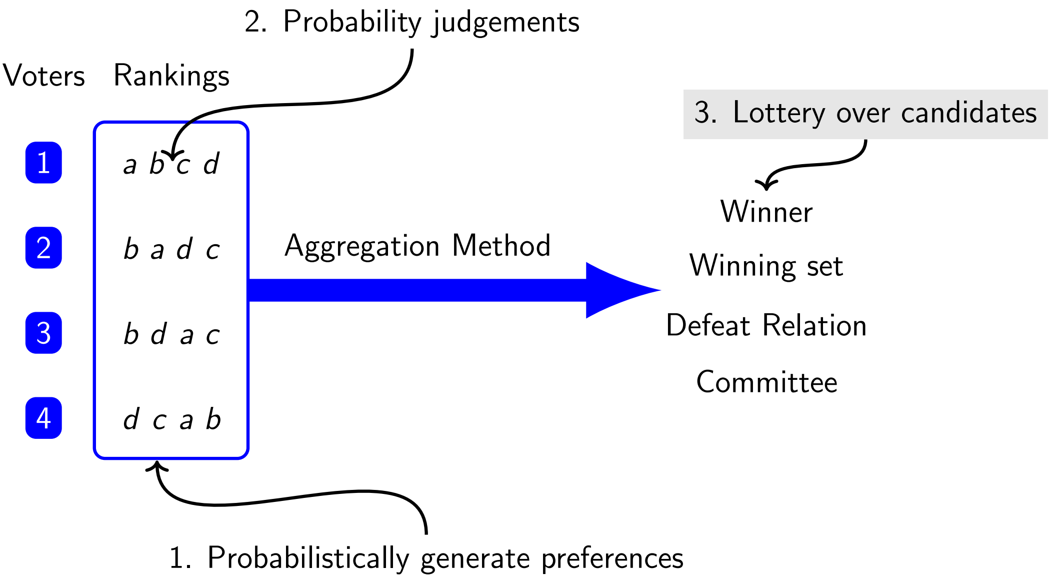
2. Probability judgements
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Rankings
3. Lottery over candidates
1
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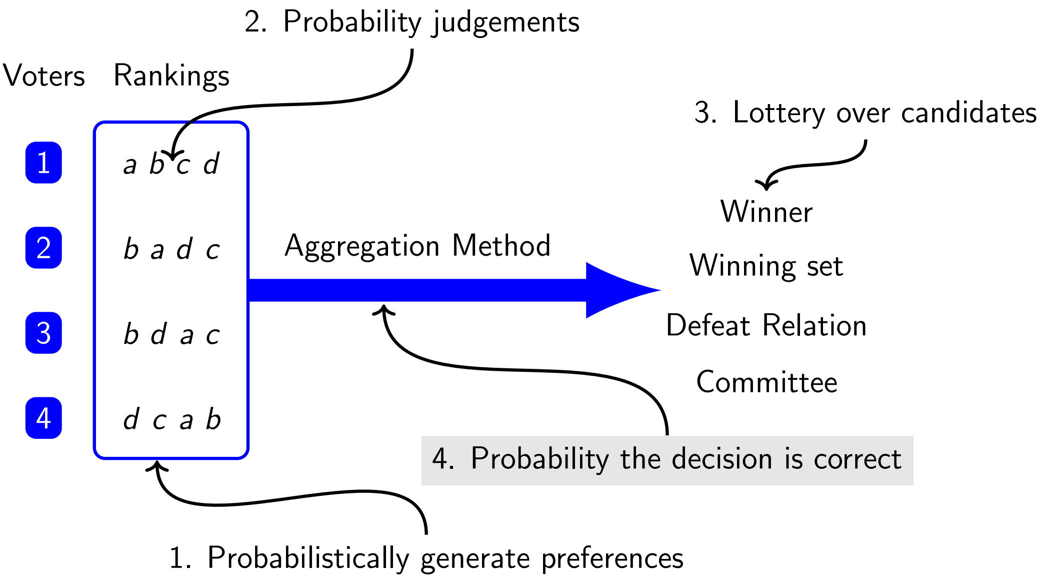

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b d a c
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d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee
4. Probability the decision is correct
1. Probabilistically generate preferences

F. Brandt. *Rolling the Dice: Recent Results in Probabilistic Social Choice*. Handbook of Computational Social Choice, 2016.

Let V be a set of voters, X a set of m alternatives.

The set of all lotteries over X is:

$$\Delta(X) = \{p \in \mathbb{R}^X \mid p(x) \geq 0 \text{ for all } x \in X \text{ and } \sum_{x \in X} p(x) = 1\}$$

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For $p \in \Delta(X)$, $supp(p) = \{x \mid p(x) > 0\}$

p is **degenerate** when $|supp(p)| = 1$.

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We write lotteries as convex combinations of alternatives, e.g., the uniform lottery on $\{a, b\}$ where $p(a) = p(b) = 1/2$ is denoted as $1/2a + 1/2b$.

Probabilistic Voting Methods

A **probabilistic social choice function (PSCF)** is a map $F : O(X)^V \to \wp(\Delta(A)) \setminus \emptyset$ such that for all $\boldsymbol{P}$, $F(\boldsymbol{P})$ is a convex set of lotteries.

Anonymity and neutrality can be defined as usual.

Random (Serial) Dictator

Random dictatorship: A voter is picked uniformly at random and this voter's most-preferred alternative is selected. Thus, the probabilities assigned by RD are directly proportional to the number of agents who top-rank a given alternative (or, in other words, the alternative's plurality score).

Random serial dictatorship (RSD): RSD selects a permutation of the agents uniformly at random and then sequentially allows agents in the order of the permutation to narrow down the set of alternatives to their most preferred of the remaining ones.

Proportional Borda

Proportional Borda: Assign probabilities to the alternatives that are proportional to their Borda scores.

3	2
a	b
b	c
c	a

- $Plurality^{eve}$ lottery is a
- RD lottery is $\frac{3}{5}a + \frac{2}{5}b$
- $Borda^{eve}$ lottery is b
- $Borda_{pro}$ lottery is $\frac{6}{15}a + \frac{7}{15}b + \frac{2}{15}c$

Margin Matrix/Graph

49	48	3
c	a	b
a	b	c
b	c	a

$$
\begin{array}{c c}
 & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 94 & -4 \\ -94 & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix}
\end{array}
$$

$$
\begin{tikzcd}
a \arrow[rr, "94"] & & b \arrow[dl, "2"] \\
& c \arrow[ul, "4"] &
\end{tikzcd}
$$

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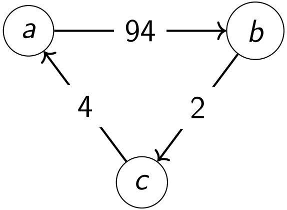

If the output of a neutral PSCF F only depends on the margin matrix/graph M, F is called **pairwise**. An advantage of pairwise PSCFs is that they are applicable even when individual preferences are incomplete or intransitive.

Maximal Lotteries

G. Kreweras. *Aggregation of preference orderings*. In Mathematics and Social Sciences I: Proceedings of the seminars of Menthon-Saint-Bernard, France (1–27 July 1960) and of Gösing, Austria (3 - 27 July 1962), pages 73 - 79, 1965.

P. C. Fishburn. *Probabilistic social choice based on simple voting comparisons*. Review of Economic Studies, 51(4):683-692, 1984.

Maximal Lotteries

H. Aziz, F. Brandl, F. Brandt, and M. Brill. *On the tradeoff between efficiency and strategyproofness*. Games and Economic Behavior, 110:1 - 18, 2018.

F. Brandl, F. Brandt, and H. G. Seedig. *Consistent probabilistic social choice*. Econometrica, 84(5):1839 - 1880, 2016.

F. Brandl, F. Brandt, M. Eberl, and C. Geist. *Proving the incompatibility of efficiency and strategyproofness via SMT solving*. Journal of the ACM, 65(2):1 - 28, 2018.

F. Brandl, F. Brandt, and C. Stricker. *An Analytical and Experimental Comparison of Maximal Lottery Schemes* . forthcoming Social Choice and Welfare, 2021.

Maximal Lotteries

M admits a (weak) Condorcet winner if M contains a nonnegative row, i.e., there is a standard unit vector v such that

$$v^T M \geq 0$$

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1	1	1
a	b	c
b	a	a
c	c	b

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1	1	1
a	b	c
b	a	a
c	c	b

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \begin{pmatrix}0 & 1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 1\end{pmatrix} \geq 0$$

Maximal Lotteries

A lottery p is maximal if $p^T M \geq 0$:

- randomized Condorcet winner
- p is "at least as good" as any other lottery:

  the expected number of agents who prefer the alternative returned by p to that returned by q is at least as large as the expected number of agents who prefer the outcome returned by q to that returned by p

Maximal Lotteries

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1	1	1
a	b	c
b	c	a
c	a	b

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\end{pmatrix} < 0$$

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a	b	c
b	c	a
c	a	b

$$\begin{pmatrix}0\\1\\0\end{pmatrix}^T \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0 & 1\end{pmatrix} < 0$$

Maximal Lotteries

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1	1	1
a	b	c
b	c	a
c	a	b

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} < 0$$

Maximal Lotteries

A lottery p is maximal if $p^T M \geq 0$:

1	1	1
a	b	c
b	c	a
c	a	b

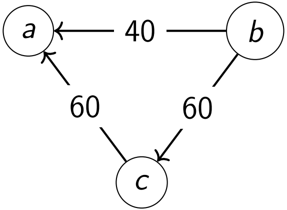
$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}^T \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \geq 0$$

Maximal Lotteries

- always exist due to the Von Neumann Minimax Theorem.
- almost always unique
- set of profiles with multiple maximal lotteries has measure zero
- always unique for odd number of voters with strict preferences
- does not require asymmetry, completeness, or even transitivity of individual preferences

20	20	60
c	a	b
a	b	c
b	c	a

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & -40 & -60 \\ 40 & 0 & 60 \\ 60 & -60 & 0 \end{pmatrix} \end{array}$$



$$\begin{array}{c c c} & \begin{array}{ccc} a & b & c \end{array} & \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0, 0 & -40, 40 & -60, 60 \\ 40, -40 & 0, 0 & 60, -60 \\ 60, -60 & -60, 60 & 0, 0 \end{pmatrix} & \begin{array}{c} \color{blue}{-60} \\ \color{blue}{40} \\ \color{blue}{-60} \end{array} \\ & \begin{array}{ccc} \color{blue}{-60} & \color{blue}{40} & \color{blue}{-60} \end{array} & \end{array}$$

Nash equilibrium: (b, b)

49	48	3
c	a	b
a	b	c
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$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 94 & -4 \\ -94 & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix} \end{array}$$

$$\begin{tikzcd} a \arrow[rr, "94"] & & b \arrow[dl, "2"] \\ & c \arrow[ul, "4"] & \end{tikzcd}$$

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0,0 & 94,-94 & -4,4 \\ -94,94 & 0,0 & 2,-2 \\ 4,-4 & -2,2 & 0,0 \end{pmatrix} \end{array}$$

There is no *pure strategy Nash equilibrium*

There is a mixed Nash equilibrium: $2/100a + 4/100b + 94/100c$.

Maximal Lotteries can be efficiently computed via linear programming

`https://voting.ml`

Population-Consistency

Population-Consistency: Whenever two disjoint electorates agree on a lottery, this lottery should also be chosen by the union of both electorates.

1	1
a	b
b	c
c	a

$\boldsymbol{P}$

$\frac{1}{2}a + \frac{1}{2}b$

1	1
a	b
c	c
b	a

$\boldsymbol{Q}$

$\frac{1}{2}a + \frac{1}{2}b$

1	1	2
a	a	b
b	c	c
c	b	a

$\boldsymbol{P} + \boldsymbol{Q}$

$\frac{1}{2}a + \frac{1}{2}b$

Saari's argument, Balinski and Laraki (2010, pg. 77); Zwicker (2016, Proposition 2.5): Multiple districts paradox, f *cancels properly*.

2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c

- no Condorcet winner in the left profile
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

- Population-consistency: $f(\boldsymbol{P}) \cap f(\boldsymbol{Q}) \subseteq f(\boldsymbol{P} + \boldsymbol{Q})$
  - Reinforcement/Consistency: if $f(\boldsymbol{P}) \cap f(\boldsymbol{Q}) \neq \varnothing$, then $f(\boldsymbol{P}) \cap f(\boldsymbol{Q}) = f(\boldsymbol{P} + \boldsymbol{Q})$

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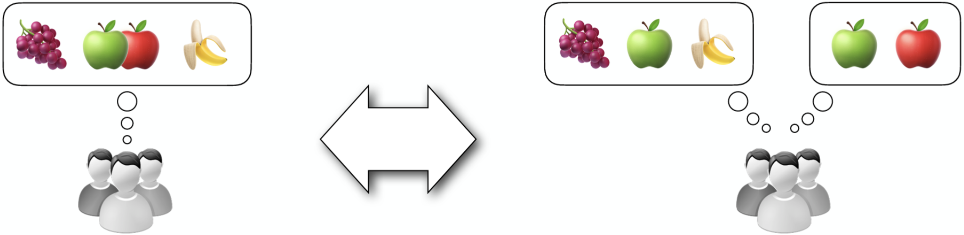
- Maximal Lotteries, Random Dictatorship, and Proportional Borda all satisfy population-consistency

Composition-Consistency



Composition-Consistency

Composition-Consistency: Decomposable preference profiles are treated component-wise. (In particular, alternatives are not affected by the cloning of other alternatives)

2	1	3
a	a	b
b'	b	b'
b	b'	a

P

$\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}b'$

3	3
a	b
b	a

$P_{|\{a,b\}}$

$\frac{1}{2}a + \frac{1}{2}b$

2	4
b'	b
b	b'

$P_{|\{b,b'\}}$

$\frac{2}{3}b + \frac{1}{3}b'$

- Many Condorcet extensions satisfy composition-consistency.
- No Pareto-optimal scoring rule satisfies composition-consistency.
- Random Dictatorship, Proportional Borda do not satisfy composition-consistency
- Maximal Lottery satisfies composition-consistency

Theorem (Brandl et al. 2016) Population-consistency and composition-consistency are incompatible in non-probabilistic social choice.

Theorem (Brandl et al. 2016) A probabilisitic social choice function satisfies population-consistency and compositional consistency if, and only if, it returns all maximal lotteries.

F. Brandl, F. Brandt, and H. G. Seedig. *Consistent probabilistic social choice*. Econometrica, 84(5):1839-1880, 2016.

Efficiency, Strategyproofness

Pareto Efficiency: no voter can be made better off without making another voter worse off.

Strategyproofness: no voter can be made better off by misrepresenting their preferences.

To define these, we need assumptions about how voters rank lotteries.

Suppose that $\succeq_i$ is voter i's weak preference relation.

$p \succeq_i^{DD'} q$ if and only if $x \succ_i y$ for all $x \in supp(p)$ and $y \in supp(q)$

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Suppose that $a \succ_i b \succ_i c$. Then,
$2/3a + 1/3b \succ_i^{DD'} c$
$2/3a + 1/3b \succ_i^{DD} 1/2b + 1/2c$

Bilinear Dominance

$p \succeq_i^{BD} q$ if and only if $p(x)q(y) \geq p(y)q(x)$ for all $x, y \in X$ with $x \succ_i y$

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Suppose that $a \succ_i b \succ_i c$. Then,
$1/2a + 1/2b \succ_i^{BD} 1/3a + 1/3b + 1/3c$

(Fishburn 1984): p bilinearly dominates q iff p is preferable to q for every skew-symmetric bilinear (SSB) utility function consistent with $\succeq_i$

Stochastic Dominance

$p \succeq_i^{SD} q$ if and only if $\sum_{y \succeq_i x} p(y) \geq \sum_{y \succeq_i x} q(y)$ for all $x \in X$

Stochastic Dominance

$p \succeq_i^{SD} q$ if and only if $\sum_{y \succeq_i x} p(y) \geq \sum_{y \succeq_i x} q(y)$ for all $x \in X$

Suppose that $a \succ_i b \succ_i c$. Then,
$1/2a + 1/2c \succ_i^{SD} 1/2b + 1/2c$

p stochastically dominates q iff p is preferable to q for every von Neumann-Morgenstern utility function consistent with $\succeq_i$

A lottery p is *SD*-**efficient** for a preference profile $\boldsymbol{R}$ if there is no lottery $q \in \Delta(X)$ such that $q \succeq_i^{SD} p$ for all i and $q \succ_j^{SD} p$ for some j.

strong *SD*-strategyproofness requires that every misreported preference relation of a voter will result in a lottery q such that $p \succeq^{SD} q$.

SD-strategyproofness: no agent can misreport his preferences to obtain a lottery q such that $q \succ^{SD} p$.

(Similar definitions for other preferences over lotteries).

ex post efficient: whenever there are alternatives x and y such that $x \succeq_i y$ for all i and $x \succ_j y$ for some j, then y should receive probability 0.

Theorem (Gibbard, 1977). Random Dictatorship is the only anonymous, strongly *SD*-strategyproof, and ex post efficient probabilistic social choice function when preferences are strict.

Theorem (Fishburn, 1984) *ML* is *ex post efficient*.

Theorem (Brandl et al., 2016). There is no anonymous, neutral, *SD*-efficient, and *SD*-strategyproof probabilistic social choice function when there are at least 4 voters and at least 4 candidates.

F. Brandl, F. Brandt, and C. Geist. *Proving the incompatibility of efficiency and strategyproofness via SMT solving*. Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI), pages 116–122. AAAI Press, 2016.

F. Brandl, F. Brandt, and C. Stricker. *An analytical and experimental comparison of maximal lottery schemes*. Social Choice and Welfare, 58(1):5–38, 2022.

Maximal Lottery Schemes

Every **maximal lottery scheme** is based on an odd and monotone function $\tau : \mathbb{Z} \to \mathbb{R}$ with $\tau(1) = 1$.

$$ML^{\tau}(\boldsymbol{R}) = \{p \in \Delta(X) \mid \sum_{x,y \in X} p(x)q(y)\tau(m_{xy}) \geq 0 \text{ for all } q \in \Delta(X)\}$$

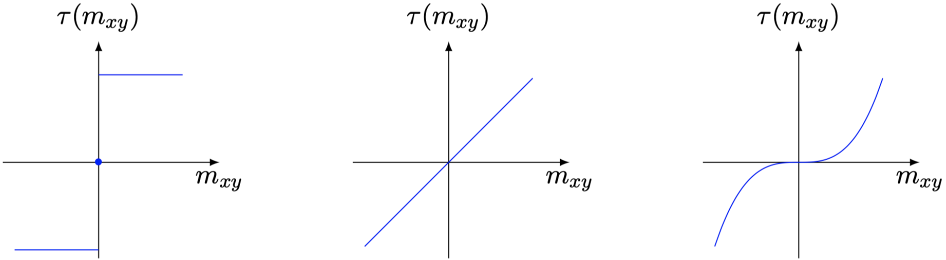

Figure 2: Different examples for τ (extended to $\mathbb{R}$): sign function, identity function, and cubic function.

$C1$-ML: ML schemes based on the sign function.

$C2$-ML: ML schemes based on the identity function.

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$C2$-ML: ML schemes based on the identity function.

49	48	3
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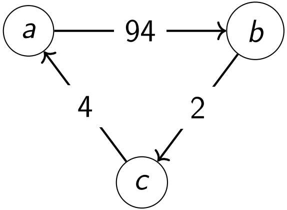

$$C1\text{-}ML\text{: } 1/3a + 1/3b + 1/3c$$

$$C2\text{-}ML\text{: } 2/100a + 4/100b + 94/100c$$

Theorem (Brandl, Brandt and Stricker) For any pair of ML schemes ML^{τ} and ML^{σ}, there is a preference profile $\boldsymbol{R}$ such that $ML^{\tau}(\boldsymbol{R}) = \{p\}$ and $ML^{\sigma}(\boldsymbol{R}) = \{q\}$ and $supp(p) \cap supp(q) = \varnothing$.

(cf. B. Dutta and J.-F. Laslier. Comparison functions and choice correspondences. Social Choice and Welfare, 16(4):513-532, 1999)

An SDS is **homogenous** when replacing every voter with a fixed number of identical clones (i.e., voters with the same preferences) does not change the outcome.

Theorem (Brandl, Brandt and Stricker) ML^{τ} is homogenous if and only if τ is based on τ' where there is a $t \geq 0$ such that $\tau'(k) = k^t$ for all $k \in \mathbb{N}$.

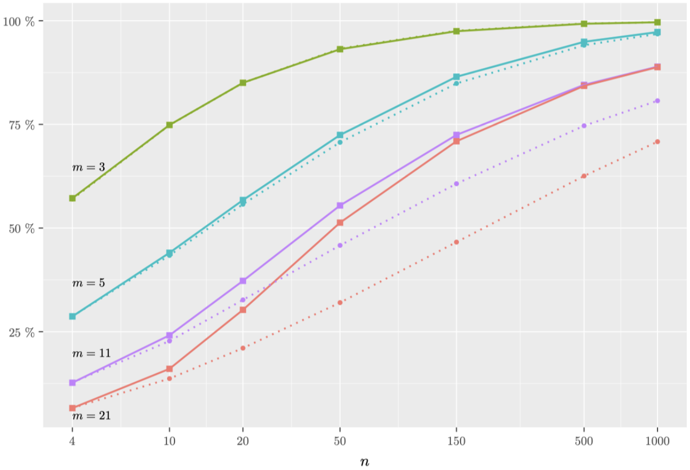

Figure 7: Frequencies of profiles R for which $|C2\text{-}ML(R)| = 1$ (solid lines) and $|C1\text{-}ML(R)| = 1$ (dotted lines), respectively. Each point is based on 100 000 samples according to the IAC model.

Figure 8: Distributions of the support sizes of *C1-ML* (left) and *C2-ML* (right) based on 100 000 samples according to the IAC model for every combination of parameters. The bars represent the frequency of a support with size $1, 3, 5$, and 7 or more stacked from bottom to top. Frequencies lower than 4% are not labeled.

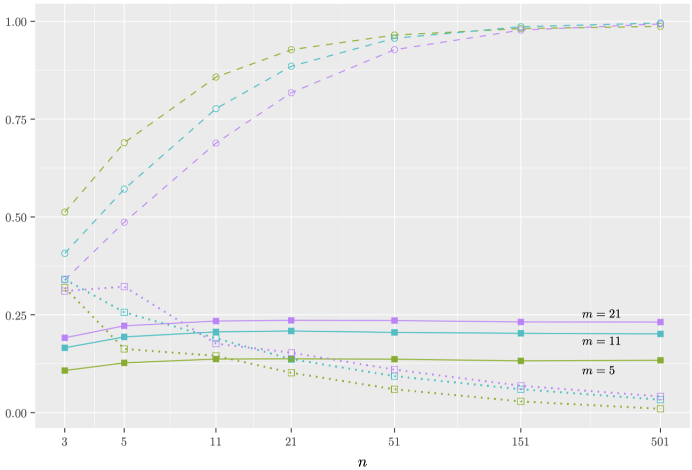

Figure 10: Average normalized Shannon entropies of *C2-ML* (solid line) compared to random dictatorship (dashed line) and the plurality rule (dotted line). Each point is based on 100 000 samples according to the IAC model.

F. Brandl and F. Brandt. *A Natural Adaptive Process for Collective Decision-Making*. manuscript, 2021.

Consider an urn filled with balls, each labeled with one of several possible collective decisions. Now, draw two balls from the urn, let a random voter pick her more preferred as the alternative, relabel the losing ball with the collective decision, put both balls back into the urn, and repeat. In order to prevent the permanent disappearance of some types of balls, once in a while, a randomly drawn ball is labeled with a random alternative.

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Brandl and Brandt prove that this process will almost surely converge to the $C2$-ML.

300	300	300
$A1$	$A1$	$A2$
$A2$	$A3$	$A3$
$A3$	$A2$	$A1$

300	300	300
$A1$	$A2$	$A3$
$A2$	$A3$	$A1$
$A3$	$A1$	$A2$