Handout 3: Social Choice Lotteries

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Majority Rule with Lotteries on Alternatives by Zeckhauser

Example: The 101 Club must choose a single form of entertainment for all club members. The membership rolls contain fifty football fanatics, fifty ballet aficionados, and a single lover of musical comedy. For the footballers the musical is almost as bad as the ballet. For the ballet enthusiasts the musical is little better than football.

50	50	1
F	B	M
M	M	B F
B	F	

M is the winner by Majority Rule without lotteries.

- If lotteries were permitted, a fifty-fifty football-ballet lottery would defeat the musical by any required plurality up to 100 out of 101.

"A political system, such as the one in the United States, which rules out lotteries might lead to dominance by the center, even if the right and left could put together a majority coalition that would prefer a lottery on the extremes to a middle outcome."

Acceptable Social Choice Lotteries by Fishburn

Setup

- Let X be a nonempty set of mutually exclusive decision alternatives that are viewed as the basic objects of choice in a group decision process.
- A **lottery** on X is a simple probability distribution on X such that $p(x) = 0$ on all but a finite number of elements from X and $\sum p(x) = 1$.
- If lottery p is used to make the social choice, then a random device that assigns probability $p(x)$ to each x for which $p(x) > 0$ is actuated to determine a 'winner'.
- Although elements in X may present aspects of risk or uncertainty to the individuals in the group, as when each $x \in X$ is a risky investment opportunity or a political candidate whose positions on salient issues are ambiguous, the probabilities used in the social choice lotteries need not refer directly to these risks or uncertainties.
  - E.g., if issue positions are interpreted as the basic alternatives, if each candidate in effect presents a lottery over issue positions to the electorate, and if the winner's actual positions and resultant policies are viewed as being selected according to his or her issue-position lottery, then the vote between the candidates can be taken to be a vote between social choice lotteries on issue positions.

- With $A \subseteq X$ the **feasible set of alternatives**, the social choice must be made from A; hence a social choice lottery will be admissible in the A context only if $p(x) = 0$ for each $x \notin A$.
- the only information from individuals about alternatives that will be used in the choice process is information about the feasible alternatives in A. This information may involve ballots and/or aspects of individuals' preferences on the single elements in A or on subsets of A or on lotteries on A.

Example: Consider a panel or jury of n judges ($i = 1, \ldots, n$) that is to award a 'prize' to one of the m contestants ($j = 1, \ldots, m$) who have qualified for a certain competition (e.g., a jury trial, primary election, sports tournament...). Let a_j denote the decision to award the 'prize' to contestant j. So, $A = \{a_1, \ldots, a_m\}$.

1. each judge submit a ballot that amounts to a lottery on A. The probability $p^i(a_j)$ that judge i assigns to a_j might be interpreted as judge i's subjective probability that contestant j is the best contestant or the one who most deserves the prize. Given a profile $(p^1, \ldots, p^n)$:
   - $p(a_j) = [p^1(a_J 1) + \cdots + p^n(a_j)]/n$
   - $p(a_j) = 1/N$ where N is the number of j such that $\sum_i p^i(a_j) = \max\{\sum_i p^i(a_1), \ldots, \sum_i p^i(a_m)\}$; and $p(a_j) = 0$ for the other candidates.
2. Ask the judges to to vote for 1 contestant.
   - $p(a_j) = n_j/n$ where n_j is the number of judges that vote for j and $n = n_1 + \cdots + n_m$.
   - $p(a_j) = n_j/N$ where N is the number of j such that $n_j = \max\{n_1, \ldots, n_m\}$; and $p(a_j) = 0$ for all other candidates.

When are lotteries acceptable?

The potential use of social choice lotteries does little to alleviate, and indeed may aggravate, the problems that can arise in social choice theories where lotteries are not permitted. Nevertheless, there are situations in which lotteries seem quite natural and are used as a matter of course in arriving at social choices.
One of the striking features of many social choice procedures is the lengths to which they will go to avoid the use of a nondegenerate social choice lottery in helping to determine the outcome of the process.

- Doctrine of *freedom and moral responsibility* of people to shape and control their destiny. According to the freedom-responsibility doctrine, the use of a lottery to make a social decision subverts a person's control over their own affairs, denies their proper role as a moral agent responsible for the health and improvement of the social organism, and otherwise constitutes a step backwards into the dark ages by relegating the decision to blind chance.
- Doctrine of *egalitarianism* or the belief in human equality in social, political and economic spheres.

When should lotteries be used

"I know of no situation in which prizes are awarded by a sanctioned social choice lottery that does not have [the following] characteristics, it is tempting to presume that they represent current minimal standards for acceptability of the use of a social choice lottery."

1. There is a set of two or more qualifying agents, such as teams, ticket holders, or citizens or residents of a certain jurisdiction who meet prescribed criteria.
2. There is a 'prize' or a set of similar prizes (kickoff option, money, invitations to jury duty, positions on the ballot) to be awarded to the qualifying agents.
3. All agents have more or less uniform attitudes towards the desirability of each prize: both teams would like the kickoff option, everyone would like to win the sweepstakes or have his name first on the ballot, and most potential draftees would probably prefer not to be drafted.
4. Each agent would acknowledge that all agents have a more or less equal claim on or right to each prize.
5. The agents do not actively compete to convince other agents or interested parties that they are more deserving of or better qualified to be awarded the prize(s).

Compare a political election (say between 2 candidates) and the selection of people to serve on a jury or to be drafted.

- Because the candidates actively try to convince the electorate of their qualifications, this situation does not satisfy the fifth characteristic.
- In the jury selection and draft lottery cases, it is frequently held that all people who meet certain basic conditions have a duty to serve on a jury or to bear arms and, moreover, that each eligible person is able to do the required job. Hence there is no provision in the underlying philosophy of these situations for agents to compete actively for the positions although people sometimes do things (which others may find morally reprehensible) to disqualify themselves from consideration. Acts of this sort do of course challenge the passivity feature of the fifth characteristic to a degree.

When one prize is desired by all qualifying agents, when all recognize an equal claim to it, and when the award is not to be based on merit or superior qualifications, what fairer and more equitable way is there of awarding it than to use an even-chance lottery?

- Even-chance lotteries are easier to comprehend and to abide by its outcome.
- People have great difficulty understanding the probabilistic aspects of lotteries with unequal chances and may be quite averse to their use.

Consider the two-candidate political situation.

- A simple coin flip to determine a winner without any vote is ruled out not only by the freedom-responsibility rule but also by the egalitarian principle as it applies to potential voters.
- 'random dictatorship': a proportional lottery rule implemented by placing all marked ballots in a drum, drawing one ballot at random, and declaring the winner to be the candidate whose name is marked on the drawn ballot.The proportional lottery rule is an even-chance lottery for voters since each voter has the same chance of naming the winner, and it clearly satisfies the egalitarian principle with regard to voters.

- The proportional lottery rule clearly clashes with the freedom-responsibility principle, which takes precedence in this situation and prescribes a nonrandom selection procedure. The egalitarian principle then enters the picture secondarily by prescribing a simple majority election in which each vote counts equally in a nonprobabilistic sense.

Social Choice using Lotteries

- x is a **Condorcet alternative** if and only if more agents prefer x to y than prefer y to x for each $y \neq x$ in A
- The **Condorcet principle**, whose congruence with the freedom-responsibility and egalitarian principles is often taken to be self-evident, says that the Condorcet alternative should be the social choice whenever there is such an alternative.
- x is a **Pareto alternative** if and only if there is no y in A that every agent prefers to x.
- The **Pareto principle**, which is consistent with our other principles, says that the social choice should be a Pareto alternative.
- The Condorcet and Pareto notions generalize in an obvious way to social choice lotteries on A.
  - p is the **Condorcet lottery** if and only if more agents prefer p to q than prefer q to p for each lottery $q \neq p$
  - p is a **Pareto lottery** if and only if there is no lottery q that every agent prefers to p.
- Let p^* represent the even-chance lottery on A.

Condorcet cycle

Let $A = \{a_1, a_2, a_3\}$ be the set of feasible alternatives with a_i the decision to award the prize to agent i.

Agent 1	Agent 2	Agent 3
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

For convenience let 0 and 1 be each agent's utility for his least and most desired alternatives, respectively, and let $u_i \in (0, 1)$ be the von Neumann-Morgenstern utility of agent i for his intermediate alternative.

- Agent 1: $p_1 + p_2 u_1$
- Agent 2: $p_2 + p_3 u_3$
- Agent 3: $p_3 + p_1 u_2$
- There is no Condorcet lottery: Shepsle [26] notes that there is a lottery p that has a simple majority over each of the three basic alternatives if and only if its expected utility for each agent exceeds the agent's u_i value. However, when such a lottery exists there must be another lottery that has a simple majority over the first.

- Every lottery is Pareto: Consider lotteries p and q and let $d_i = p_i - q_i$. We have that $d_1 + d_2 + d_3 = 0$. All agents will prefer p to q if and only if

$$d_1 + d_2 u_1 > 0$$
$$d_2 + d_3 u_3 > 0$$
$$d_3 + d_1 u_2 > 0$$

1. If $d_i = 0$ for some i then since $d_j + d_k = 0$ fo the other two, one of the three inequalities must fail
2. If $d_i \neq 0$ for all i, then the inequalities and $\sum d_i = 0$ require $d_i > 0$ for exactly two of the three d_i, say d_1 and d_2 with $d_3 = -(d_1 + d_2)$ and in this case the third inequality is violated since $-(d_1 + d_2) + d_1 u_3 < 0$.

Since neither the Condorcet principle nor the Pareto principle (applied either to basic alternatives or to lotteries) helps in any way to discriminate among lotteries, an even-chance lottery on A seems natural in this situation especially in view of the symmetry aspect of the basic preferences.

Adding a compromise alternative

Let c be the decision to award the compromise prize (e.g., the prize could be split into three equal parts).

Agent 1	Agent 2	Agent 3
a_1	a_2	a_3
c	c	c
a_2	a_3	a_1
a_3	a_1	a_2

- c is the Condorcet alternative in $\{a_1, a_2, a_3, c\}$
- the award of c seems reasonable and equitable;
- the presence of c may ease our minds about the possibility of making the award by blind chance.

However, a closer look may reveal a disquieting possibility. In particular, every agent might prefer the even-chance lottery p^* on $\{a_1, a_2, a_3\}$ to the compromise alternative c.

- Except perhaps for the aspect of chance, the choice of the even-chance lottery seems wholly consistent with the freedom-responsibility and egalitarian principles.
- On the other hand, one might argue that the Condorcet alternative c is the better choice since more agents will prefer c to the lottery outcome after the fact.
- If the latter position is taken, and if it is believed to be most in line with the two basic principles, then it seems necessary to demonstrate why the presumably morally responsible and rational agents would be ill-advised to implement an option that they uniformly prefer to the Condorcet alternative even though each knows that the risky option is more likely than not to leave him in a less preferred position.

Example. Suppose that there are 4 alternatives $\{a, b, c, d\}$ and three members of the group feel as follows:

1. a is terrific; d is all right and is slightly better than b and c which are satisfactory;
2. b is terrific; d is all right and is slightly better than a and c which are satisfactory;
3. c is terrific; d is all right and is slightly better than a and b which are satisfactory;

Each member of the group would prefer an even-chance lottery on $\{a, b, c\}$ to the Condorcet winner d.

- The Condorcet winner is Pareto-dominated by the even-chance lottery on $\{a, b, c\}$
- The even-chance lottery on $\{a, b, c\}$ is Pareto-optimal
- There are other lotteries that are majority preferred to the even-chance lottery on $\{a, b, c\}$ (e.g., the even-chance lottery on $\{a, b\}$ is preferred by 2 out of 3 voters).

Theorem (Fishburn 1972). Suppose that each each voters preferences on the set of lotteries satisfies the *weak individual axiom* and that the lottery p is majority preferred to every lottery q such that $p \neq q$. Then p must be the degenerate lottery for some alternative.

P. Fishburn, Lotteries and Social Choices, Journal of Economic Theory, 5, pp. 189 - 207, 1972.

Multiple-prize situations

Let C_1, C_2, and C_3 be three candidates competing in an election. The decision to be made concerns the order of their names on the ballot. Let $C_iC_jC_k$ denote the ballot on which C_i's name is first, C_j's name is second, and C_k's name is last. The six ballot-position arrangements will be denoted as a l through a6 where

$a_1 = C_1C_2C_3$

$a_2 = C_1C_3C_2$

$a_3 = C_2C_1C_3$

$a_4 = C_2C_3C_1$

$a_5 = C_3C_1C_2$

$a_6 = C_3C_2C_1$

Assuming that each candidate's preferences are governed solely by his position on the ballot, the following preference orders can be expected:

C_1		C_2		C_3	
a_1	a_2	a_3	a_4	a_5	a_6
a_3	a_5	a_1	a_6	a_2	a_4
a_4	a_6	a_2	a_5	a_1	a_3

With $p_1 = p(a_i)$, the expected utilities of the candidates for lottery p on $A = \{a_1, \ldots, a_6\}$ are: (where $u_i \in (0, 1)$ is the von Neumann-Morgnester utility for agent i of the second ranked alternative)

C_1: $(p_1 + p_2) + (p_3 + p_5)u_1$

C_2: $(p_3 + p_4) + (p_1 + p_6)u_2$

C_3: $(p_5 + p_6) + (p_2 + p_4)u_3$

Theorem 1. Suppose that p^* is the even-chance lottery such that $p^*_i = \frac{1}{6}$. Lottery p^* is a Pareto lottery if and only if $u_1 = u_2 = u_3$.

Since it would be unusual at the very least to attempt to determine the u_i in an actual ballot situation, and since the egalitarian principle might be interpreted as allowing only p^* as a potentially acceptable lottery, the preceding theorem might be taken to be little more than a technical curiosity. Nevertheless, it does suggest that in cases where more general lotteries might be considered, the 'obviously equitable' even-chance lottery may be unanimously less preferred than some other lottery.

- There is a natural compromise alternative c which directs that each of the six a_i orders appears on about one-sixth of the ballots.
- If every candidate prefers c to his intermediate alternatives, such as $(a_1\ a_2)\ c\ (a_3\ a_5)(a_4\ a_6)$ for C_1, then c will be the Condorcet alternative.
- But, it could also be true that every candidate prefers the even chance lottery p^* on $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ to c. However, p^* in this case will itself be unanimously less preferred than some other lottery if the u_i are not all equal.

One indivisible prize

Assume that there is one indivisible and uniformly desirable prize to be awarded to n agents, with a_i the decision to award the prize to i. Consider the situation in which each agent wants the prize but doesn't care who gets the prize if the agent doesn't get it.

Agent 1: $a_1\ (a_2\ a_3\ \cdots\ a_n)$

Agent 2: $a_2\ (a_1\ a_3\ \cdots\ a_n)$

$\vdots$

Agent n: $a_n\ (a_1\ a_2\ \cdots\ a_{n-1})$

- All lotteries on $A = \{a_1, \ldots, a_n\}$ Pareto lotteries
- For every lottery p there is another lottery that is preferred to the first by $n - 1$ of the n voters: by reducing one $p_j > 0$ and increasing every p_i or $i \neq j$, we obtain a new lottery that everyone except j prefers to p.
- The egalitarian principle supports the even-chance lottery p^*

- p^* is the maximin lottery since it uniquely maximizes the minimum, over agents, of the ratio of the utility of the lottery minus the utility of the worst alternative to the utility of the best minus the utility of the worst.
- The unique lottery that is least likely to be defeated $n-1$ to 1 by another lottery chosen at random by the uniform distribution over the lottery space is p^*.

J. Broome, Fairness, Proceedings of the Aristotelian Society, 1990 - 1991, New Series, Vol. 91 (1990 - 1991), pp. 87-101.