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Formal utilitarianism and range voting ☆

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HIGHLIGHTS

- In formal utilitarian voting, voters assign a real number to each alternative.
- The alternative with the highest total score wins.
- Range voting is the same, only each score must be between zero and one.
- We give axiomatic characterizations via reinforcement and overwhelming majority.
- We also use two new axioms: maximal expressiveness, and "no minority overrides".

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ABSTRACT

In *formal utilitarian* voting, each voter assigns a numerical value to each alternative, and society chooses the alternative with the highest total value. *Range voting* is similar, except that each voter's values are constrained to lie in the interval [0, 1]. We characterize these rules via the axioms of anonymity, neutrality, reinforcement, overwhelming majority, and two novel conditions: maximal expressiveness, and an absence of "minority overrides".

1. Introduction

Classical utilitarianism is based on the principle that social decisions should be made so as to maximize the "net benefit" to the society, where this "net benefit" is the sum of the benefits accruing to all members of that society. In modern welfare economics, this principle is embodied by the utilitarian social welfare function, which has a variety of appealing axiomatic characterizations (Blackorby et al., 2002). However, these axiomatizations assume some form of interpersonally comparable cardinal utility, which is regarded by many people as being metaphysically dubious, or at least, highly impractical.

In the setting of Arrovian preference aggregation, "positional" rules such as the Borda rule are attractive in part because they appeal to the same "net benefit" intuition as utilitarianism, but they assume only ordinal and noncomparable preferences. But these rules effectively *impose* a cardinal utility representation, by assigning a numerical value to each position in each agent's preference order. This cardinalization might not be appropriate in all cases. For example, in one social decision, a voter Alice might have a very mild preference $w \succ x$, and a very mild preference $y \succ z$, but a very strong preference for either of w or x over either of y or z. In another social decision, Alice might have a clear favorite alternative w', which she strongly prefers over any of x', y', or z', and relatively weak preferences $x' \succ y' \succ z'$. But the Borda rule (for example) does not recognize such nuances; it will impute to Alice the utility function $u(w) = u(w') = 3$, $u(x) = u(x') = 2$, $u(y) = u(y') = 1$ and $u(z) = u(z') = 0$. As long as we remain within the framework of Arrovian preference aggregation, Alice has no way of expressing the relative intensity of different preferences.

Once we leave the Arrovian framework, other options become available. One option is *formal utilitarianism*, where each person votes by assigning a numerical 'score' to each alternative, and society picks the alternative with the highest average score.[1] However, this rule is obviously vulnerable to strategic preference exaggeration, unless we impose upper and lower bounds on the scores that voters are allowed to assign. Once we impose such limits, we have a voting rule called *range voting* (Smith, 2000; Gaertner and Xu, 2012; Macé, 2013).[2]

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[1] 'Formal' utilitarian voting corresponds to the *true* utilitarian social welfare order only if the scores assigned by each voter are given by her cardinal utility function. But these scores could also be some monotone transform of her cardinal utility function (e.g. the Nash SWO is obtained by adding the logarithms of voters' utilities). Or these scores could be completely unrelated to cardinal utility data. Hence the qualifier 'formal'.

[2] Range voting is very similar to the *relative utilitarian* social choice function introduced by Dhillon (1998) and Dhillon and Mertens (1999). But there are two subtle differences. First, Dhillon and Mertens interpret each voter's preference

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This note will characterize formal utilitarianism and range voting, using the axioms of anonymity, neutrality, reinforcement, and overwhelming majority. The first two axioms are standard. *Reinforcement* means that, if two disjoint subpopulations both independently select the same alternative, then the union of these two subpopulations should also select that alternative; this condition appears frequently in axiomatic characterizations of positional preference aggregation rules,[3] and abstract scoring rules,[4] as well as characterizations of the Borda rule,[5] approval voting rule,[6] and the relative utilitarian rule.[7] *Overwhelming majority* is an Archimedean or continuity condition, which means that a sufficiently large subpopulation of voters can determine the outcome and "overwhelm" any small minority; it also appears in much of the aforementioned literature. I will also impose two novel conditions: *maximal expressiveness*, and *no minority overrides*. The first of these captures the idea that Alice should be able to express the difference in the intensity of her preferences $w \succ x \succ y \succ z$, compared to the intensity of her preferences $w' \succ x' \succ y' \succ z'$. (We do not need to take a specific position on exactly what such preference intensities *mean*, operationally speaking, or how they should be interpersonally compared—the claim is merely that voters should have maximal expressive freedom.) The second condition captures the idea that Alice should not be able to single-handedly control the outcome of the election by claiming that, for her, $u(w) - u(z) = 10^{100}$.

Theorem 1 (below) states that formal utilitarianism is the most expressive variable-population voting rule satisfying the axioms of anonymity, neutrality, reinforcement, and overwhelming majority. Theorem 2 states that range voting is the most expressive rule if we impose, in addition, the requirement of no minority overrides. The proofs of these theorems depend on results and concepts introduced in Pivato (2013). That paper studies two abstract classes of voting rules: *balance rules* and *scoring rules*. The present paper first uses results from Pivato (2013) to prove a key technical result, Proposition A.1 (in the Appendix), which can be seen as a generalization of the main result of Myerson (1995). Proposition A.1 is then used to prove Theorems 1 and 2. The present paper can thus be seen as a concrete application of the abstract results of Pivato (2013).

The next section will formally state the axioms and main results. Appendix A contains the proofs of the main results. Appendix B discusses the independence of the axioms.

2. Model and main results

Let $\mathcal{X}$ be a finite set of social alternatives, and let $\mathcal{V}$ be a (possibly infinite) set of possible 'signals' which could be sent by each voter. Let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ and $\mathbb{Z} := \{\pm n;\ n \in \mathbb{N}\}$. For any $\mathbf{n} \in \mathbb{Z}^{\mathcal{V}}$, let $\|\mathbf{n}\| := \sum_{v \in \mathcal{V}} |n_v|$. Define $\mathbb{N}^{\langle \mathcal{V} \rangle} := \{\mathbf{n} \in \mathbb{N}^{\mathcal{V}};\ \|\mathbf{n}\| < \infty\}$. If $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, then $\mathbf{n}$ represents an anonymous profile of voters: for each $v \in \mathcal{V}$, we interpret n_v as the number of voters sending the signal v, while $\|\mathbf{n}\|$ is the size of the whole population. Note that we do not fix $\|\mathbf{n}\|$ in advance. Let $\mathbf{0} = (0, 0, \ldots, 0)$ be the all-zeros vector in $\mathbb{N}^{\langle \mathcal{V} \rangle}$. A *variable population, anonymous voting rule* is a correspondence $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ such that $F(\mathbf{0}) = \mathcal{X}$. Thus, for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, the outcome $F(\mathbf{n}) \subseteq \mathcal{X}$ is a nonempty set (typically a singleton).

This paper is concerned with two voting rules. In the *formal utilitarian* voting rule, $\mathcal{V} := \mathbb{R}^{\mathcal{X}}$. In other words, each voter's signal assigns a real-valued "score" to each alternative. We compute the total score which each alternative receives from all voters.[8] Then, the alternative(s) with the highest total score wins. *Range voting* is a very similar procedure, except that $\mathcal{V} := [0, 1]^{\mathcal{X}}$. Let us now consider the axioms which will be the basis of our characterization results.

Reinforcement. A voting rule $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ satisfies *reinforcement*[9] if the following is true: for any $\mathbf{n}, \mathbf{m} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, if $F(\mathbf{n}) \cap F(\mathbf{m}) \neq \emptyset$, then $F(\mathbf{n}+\mathbf{m}) = F(\mathbf{n}) \cap F(\mathbf{m})$. Here, the profile $(\mathbf{n}+\mathbf{m})$ represents a union of two disjoint subgroups, represented by profiles $\mathbf{n}$ and $\mathbf{m}$. Reinforcement says: if $x \in \mathcal{X}$ and both $\mathbf{n}$ and $\mathbf{m}$ endorse x (i.e. $x \in F(\mathbf{n})$ and $x \in F(\mathbf{m})$), then we should have $x \in F(\mathbf{n} + \mathbf{m})$. Furthermore, in this case, $F(\mathbf{n} + \mathbf{m})$ should consist of *only* those $x \in \mathcal{X}$ which receive this joint endorsement.

Neutrality. Let $\Pi_{\mathcal{V}}$ be the set of all permutations of $\mathcal{V}$. This set forms a group under the operation of function composition.[10] For any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ and $\pi \in \Pi_{\mathcal{V}}$, we define $\pi(\mathbf{n}) := \mathbf{m}$, where $m_v := n_{\pi^{-1}(v)}$ for all $v \in \mathcal{V}$. Let $\Pi_{\mathcal{X}}$ be the set of all permutations of $\mathcal{X}$; again it is a group under composition. A voting rule $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ is *neutral* if there exists a group homomorphism[11] $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ (the *neutralizer*) such that, for all $\pi \in \Pi_{\mathcal{X}}$, if $\widetilde{\pi} := \nu(\pi)$, then $F(\widetilde{\pi}(\mathbf{n})) = \pi(F(\mathbf{n}))$ for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$. Thus, every alternative in $\mathcal{X}$ is treated equally: for any $x, y \in \mathcal{X}$, and every profile $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ such that $x \in F(\mathbf{n})$, there exists some permutation $\mathbf{n}'$ of $\mathbf{n}$ such that $y \in F(\mathbf{n}')$.

Overwhelming majority. A voting rule F satisfies *overwhelming majority*[12] if, for any $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, there exists some $M \in \mathbb{N}$ such that, for all $m > M$, we have $F(m\mathbf{n} + \mathbf{n}') \subseteq F(\mathbf{n})$. This means: if one sub-population of voters (represented by $m\,\mathbf{n}$) is much larger than another sub-population (represented by $\mathbf{n}'$), then the choice of the combined population should be determined by the choice of the larger sub-population—except that the smaller sub-population may act as a 'tie-breaker' in some cases.

Maximal expressiveness. Let $\mathcal{V}$ and $\mathcal{W}$ be two sets of 'signals', and let $\alpha : \mathcal{V} \longrightarrow \mathcal{W}$ be a ('translation') function. Define $\alpha_* : \mathbb{N}^{\langle \mathcal{V} \rangle} \longrightarrow \mathbb{N}^{\langle \mathcal{W} \rangle}$ as follows: for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, and any $w \in \mathcal{W}$, let $\alpha_*(\mathbf{n})_w := \sum\{n_v;\ v \in \mathcal{V} \text{ and } \alpha(v) = w\}$ (in particular, if $\alpha(v) \neq w$ for all $v \in \mathcal{V}$, then $\alpha_*(\mathbf{n})_w = 0$). Given two voting rules $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ and $G : \mathbb{N}^{\langle \mathcal{W} \rangle} \rightrightarrows \mathcal{X}$, we say that G is *at least as expressive as* F if there is some function $\alpha : \mathcal{V} \longrightarrow \mathcal{W}$ such that, for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, $G[\alpha_*(\mathbf{n})] = F(\mathbf{n})$. Thus, for any $v \in \mathcal{V}$, voting for v in the rule F is effectively equivalent to voting for $\alpha(v)$ in G. Thus, the

data as a von Neumann–Morgenstern (vNM) utility function, normalized to range over the interval [0, 1]. But range voting does not propose any particular utility interpretation, either vNM or otherwise. Second, relative utilitarianism requires each voter's utility scores to span the *entire* interval [0, 1] (i.e. her minimum score must be 0, her maximum must be 1); range voting does not require this.

3 See Smith (1973) (who calls this condition 'separability') and Young (1974b, 1975) (who calls it 'consistency').

4 See Myerson (1995) or Pivato (2013).

5 See Young (1974a, 1975) and Nitzan and Rubinstein (1981).

6 See Fishburn (1978), Morkelyunas (1981), Alós-Ferrer (2006), and Alcantud and Laruelle (2013).

7 See Dhillon (1998) and Dhillon and Mertens (1999), who refer to reinforcement as 'extended Pareto'.

8 Formally, denote a generic element of $\mathcal{V}$ by $v = (v_x)_{x \in \mathcal{X}}$, where $v_x \in \mathbb{R}$ for all $x \in \mathcal{X}$. Then for any profile $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ and alternative $x \in \mathcal{X}$, the *total score* of x is $\sum_{v \in \mathcal{V}} n_v v_x$.

9 Sometimes this is called *consistency*.

10 Formally, a *group* is an ordered pair $(\mathcal{G}, *)$, where $\mathcal{G}$ is a set and $* : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ is an associative binary operation, such that (1) there exists a (unique) *identity* element $e \in \mathcal{G}$ is such that $e * g = g = g * e$ for all $g \in \mathcal{G}$, and (2) for all $g \in \mathcal{G}$, there is a (unique) *inverse* element g^{-1} such that $g^{-1} * g = e = g * g^{-1}$. Often, we will abuse notation and refer to "the group $\mathcal{G}$", when $*$, e, and the inversion operation are clear from context. See Dummit and Foote (2004) for a good introduction to group theory.

11 If $(\mathcal{G}, *)$ and $(\mathcal{G}', *')$ are two groups, then a *group homomorphism* is a function $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ such that $\phi(g_1 * g_2) = \phi(g_1) *' \phi(g_2)$ for all $g_1, g_2 \in \mathcal{G}$. It follows that $\phi(e) = e'$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in \mathcal{G}$ (Dummit and Foote, 2004). In the present example, this means that $\nu(\pi_1 \circ \pi_2) = \nu(\pi_1) \circ \nu(\pi_2)$ for all $\pi_1, \pi_2 \in \Pi_{\mathcal{X}}$.

12 Sometimes this is called *continuity* or the *Archimedean property*.

voters can express any profile of opinions via G which they could have expressed via F.

For example, if $\mathcal{V} \subset \mathcal{W}$, then $\mathbb{N}^{\langle\mathcal{V}\rangle}$ can be regarded as a subspace of $\mathbb{N}^{\langle\mathcal{W}\rangle}$. In this case, if F is just the restriction of G to $\mathbb{N}^{\langle\mathcal{V}\rangle}$, then G is at least as expressive as F (to see this, let $\alpha : \mathcal{V} \longrightarrow \mathcal{W}$ be the inclusion map). In particular, the formal utilitarian rule is at least as expressive as the range voting rule, which in turn is at least as expressive as the approval voting rule, which in turn is at least as expressive as the plurality rule. Likewise, the range voting rule is at least as expressive as the Borda rule.

Two voting rules are *equivalent* if each is at least as expressive as the other. For example, let $\mathcal{V} := \mathbb{R}^{\mathcal{X}}$ and let F_U be the formal utilitarian voting rule. Let $\mathcal{V}_+$ be the non-negative orthant of the vector space $\mathcal{V}$, and let F_+ be the restriction of F_U to $\mathbb{N}^{\langle\mathcal{V}_+\rangle}$. Then F_+ is equivalent to F_U.[13] Likewise, let $\mathcal{V}_0$ be the hyperplane of all vectors in $\mathcal{V}$ whose entries sum to zero, and let F_0 be the restriction of F_U to $\mathbb{N}^{\langle\mathcal{V}_0\rangle}$. Then F_0 is equivalent to F_U.[14]

The rule F is the *most expressive* member of some class of rules if it is at least as expressive as every other element of that class. Clearly, such a rule, if it exists, is unique up to equivalence. We now come to our first main result.

Theorem 1. *Let $\mathcal{X}$ be a finite set. Formal utilitarian voting is the most expressive $\mathcal{X}$-valued voting rule which satisfies reinforcement, neutrality, and overwhelming majority.*

Minority overrides. Unfortunately, formal utilitarian voting suffers from an obvious flaw. For any $v \in \mathcal{V}$, define $\mathbf{1}^v \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ by $(\mathbf{1}^v)_v := 1$, whereas $(\mathbf{1}^v)_w := 0$ for all $w \in \mathcal{V} \setminus \{v\}$. A voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ *admits minority overrides* if, for any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, there is some $v \in \mathcal{V}$ such that $F(\mathbf{n} + \mathbf{1}^v) \neq F(\mathbf{n})$. Thus, regardless of the size of the populace and the weight of existing public opinion, a single voter can always cast a vote which changes the outcome. Such 'overrides' not only generate political instability; they are arguably undemocratic. It might be better if F did *not* admit minority overrides.[15] If $\mathcal{V}$ is finite, then any rule satisfying overwhelming majority will not admit minority overrides.[16] However, we will be interested in the case when $\mathcal{V}$ is infinite. For neutral voting rules, an absence of minority overrides is effectively equivalent to imposing upper and lower bounds on the scores which voters can assign to alternatives (see Lemma A.8). For example: formal utilitarian voting admits minority overrides, but range voting does not. Here is our second main result.

Theorem 2. *Let $\mathcal{X}$ be a finite set. Range voting is the most expressive $\mathcal{X}$-valued voting rule which satisfies reinforcement, neutrality, overwhelming majority, and does not admit minority overrides.*

We have already noted that the most expressive voting rule in a particular class of rules is only unique "up to isomorphism". This means that it is possible that two different scoring systems both yield the maximally expressive rule. The formal utilitarian and range voting rules both use rather large sets of signals ($\mathbb{R}^{\mathcal{X}}$ and $[0, 1]^{\mathcal{X}}$, respectively). This raises the question: is there a voting rule with a much smaller (e.g. countable or finite) signal set which is equivalent to one of these rules? The next result answers this question in the negative.

Proposition 3. *There is no voting rule with a countable signal set which is as expressive as either formal utilitarianism or range voting.*

It is also easy to see that there is no maximality result comparable to Theorems 1 or 2 if we restrict ourselves to the class of voting rules with finite or countable signal sets.[17] However, for practical purposes, an "approximation" of range voting using a very large but finite set of scores is probably sufficient. (It seems unlikely that many voters really need scores with more than five decimal places of precision to express their political preferences, much less that they require irrational numbers.) Furthermore, there is a sense in which, for strategic voters, the approval voting rule is already the "maximally" expressive rule in the class of voting rules which satisfy reinforcement, neutrality, overwhelming majority, and do not admit minority overrides. Núñez and Laslier (in preparation) have shown that, in a large population of strategic voters, approval voting is *strategically equivalent* to range voting, meaning that, for a given profile of voter utility functions, the two rules will always produce the same outcomes in strategic voting equilibrium. Thus, from a strategic voting perspective, it seems that the extra expressiveness offered by range voting is redundant.

However, this judgment may be too hasty. Núñez and Laslier have also shown that, in a small population, the two rules are *not* strategically equivalent. Furthermore, many voters are not strategic; they regard voting as an act of political expression, as well as a chance to influence the outcome. For such voters, the extra expressiveness offered by range voting may be valuable for non-strategic reasons.

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Appendix A. Proofs

For any vector $\mathbf{s} = (s_v)_{v\in\mathcal{V}} \in \mathbb{R}^{\mathcal{V}}$, we define a function $\mathbf{s} : \mathbb{N}^{\langle\mathcal{V}\rangle} \longrightarrow \mathbb{R}$ by setting $\mathbf{s}(\mathbf{n}) := \sum_{v\in\mathcal{V}} n_v\, s_v$ for all $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$.[18] A *real-valued score system* on $(\mathcal{X}, \mathcal{V})$ is an $\mathcal{X}$-indexed collection of vectors $\mathsf{S} := \{\mathbf{s}^x\}_{x\in\mathcal{X}} \subset \mathbb{R}^{\mathcal{V}}$. The *scoring rule* determined by S is the voting rule $F_{\mathsf{S}} : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ defined as follows:

$$F_{\mathsf{S}}(\mathbf{n}) := \operatorname*{argmax}_{x\in\mathcal{X}}\ \mathbf{s}^x(\mathbf{n}), \quad \text{for all } \mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}. \tag{A.1}$$

Intuitively, $\mathbf{s}^x(\mathbf{n})$ is the 'score' which alternative x receives from the profile $\mathbf{n}$; each voter who sends the signal v contributes s^x_v 'points' to this score. The alternative with the highest score wins. For example, the Borda rule, the plurality rule, the formal utilitarian rule, range voting, and approval voting are all scoring rules. Myerson (1995) showed that, if $\mathcal{V}$ is finite, and $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ is neutral, and satisfies reinforcement and overwhelming majority, then F is a scoring rule.

For any $\pi \in \Pi_{\mathcal{V}}$ and any $\mathbf{r} = (r_v)_{v\in\mathcal{V}} \in \mathbb{R}^{\mathcal{V}}$, we define $\mathbf{r}\pi \in \mathbb{R}^{\mathcal{V}}$ by setting $(\mathbf{r}\pi)_v := r_{\pi(v)}$ for all $v \in \mathcal{V}$. Let $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a group homomorphism. A real-valued score system $\mathsf{S} = \{\mathbf{s}^x\}_{x\in\mathcal{X}}$ is *ν-neutral* if, for all $\pi \in \Pi_{\mathcal{X}}$ and $x, y \in \mathcal{X}$, if $\pi(y) = x$ and $\widetilde{\pi} := \nu(\pi)$, then $\mathbf{s}^x\,\widetilde{\pi} = \mathbf{s}^y$. More generally, we say S is *neutral* if it is ν-neutral for some group homomorphism $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$. For example, all the scoring rules mentioned in the previous

13 *Proof*: F_U is at least as expressive as F_+ because $\mathcal{V}_+ \subset \mathcal{V}$. To see the converse, for any $v = (v_x)_{x\in\mathcal{X}} \in \mathcal{V}$, let $\underline{v} := \min_{x\in\mathcal{X}} v_x$, and then define $\alpha(v) := (v_x - \underline{v})_{x\in\mathcal{X}} \in \mathcal{V}_+$. This defines a function $\alpha : \mathcal{V} \longrightarrow \mathcal{V}_+$. For any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, it is easy to see that $F_+[\alpha(\mathbf{n})] = F_U(\mathbf{n})$.

14 *Proof*: F_U is at least as expressive as F_0 because $\mathcal{V}_0 \subset \mathcal{V}$. To see the converse, let $\alpha : \mathcal{V} \longrightarrow \mathcal{V}_0$ be the orthogonal projection function. For any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, it is easy to see that $F_+[\alpha(\mathbf{n})] = F_U(\mathbf{n})$.

15 Of course, there will always be *some* profiles where a single voter can change the outcome; the point is that this should not be true for *all* profiles.

16 *Proof.* Find $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ such that $|F(\mathbf{n})| = 1$. If $M \in \mathbb{N}$ is large enough, then overwhelming majority protects the profile $M\mathbf{n}$ from minority overrides.

17 *Proof sketch*. Given any scoring rule F_1 with a countable signal set S, it is always possible to construct another scoring rule F_2 which is strictly more expressive, by deploying a countable signal set S_2 which is a strict superset of S_1.

18 If $\mathcal{V}$ was finite, then we could simply write $\mathbf{s}(\mathbf{n}) = \mathbf{s} \bullet \mathbf{n}$, where '$\bullet$' is the inner product operation on $\mathbb{R}^{\mathcal{V}}$.

paragraph have neutral score systems, with the obvious neutralizer homomorphisms.

A voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ is *trivial* if $F(\mathbf{n}) = \mathcal{X}$ for all $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$; otherwise F is *nontrivial*. The proofs of Theorems 1 and 2 both depend on the next result, which extends Myerson's theorem to the case when $\mathcal{V}$ is infinite.

Proposition A.1. *Let $\mathcal{X}$ be a finite set, and let $\mathcal{V}$ be an arbitrary set. If a voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies reinforcement and overwhelming majority, and is neutral and nontrivial, then $F = F_{\mathsf{S}}$, where S is a neutral, real-valued score system on $(\mathcal{X}, \mathcal{V})$.*

The proof of Proposition A.1 requires several preliminary lemmas, which build on previous results from Pivato (2013) concerning scoring rules, balance rules, and linearly ordered Abelian groups.

A *linearly ordered Abelian group* is a triple $(\mathcal{R}, +, >)$, where $\mathcal{R}$ is a set, '$+$' is an Abelian group operation,[19] and '$>$' is a complete, antisymmetric, transitive binary relation such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$. For example: the set $\mathbb{R}$ of real numbers is a linearly ordered Abelian group, with the standard ordering and addition operator. So is any subgroup of $\mathbb{R}$. For any nonzero $N \in \mathbb{N}$, let $\mathbb{R}^N_{\text{lex}}$ denote the group $\mathbb{R}^N$ with vector addition and the lexicographic order; then $\mathbb{R}^N_{\text{lex}}$ is a linearly ordered Abelian group.

For any $\mathbf{r} = (r_v)_{v\in\mathcal{V}} \in \mathcal{R}^{\mathcal{V}}$, we define a function $\mathbf{r} : \mathbb{N}^{\langle\mathcal{V}\rangle} \longrightarrow \mathcal{R}$ by setting $\mathbf{r}(\mathbf{n}) := \sum_{v\in\mathcal{V}} n_v r_v$ for all $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$. An *$\mathcal{R}$-valued balance system* on $(\mathcal{X}, \mathcal{V})$ is an $\mathcal{X}^2$-indexed collection $\mathsf{B} := \{\mathbf{b}^{x,y}\}_{x,y\in\mathcal{X}} \subset \mathcal{R}^{\mathcal{V}}$ such that $\mathbf{b}^{x,y} = -\mathbf{b}^{y,x}$ for all $x, y \in \mathcal{X}$ (in particular, $\mathbf{b}^{x,x} = 0$ for all $x \in \mathcal{X}$). We then define $F_{\mathsf{B}} : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ as follows: for all $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, we set $F_{\mathsf{B}}(\mathbf{n}) := \{x \in \mathcal{X};\ \mathbf{b}^{x,y}(\mathbf{n}) \geq 0 \text{ for all } y \in \mathcal{X}\}$ (this set might be empty). We say that F_{B} is a *perfect balance rule* if, for all $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, we have $F_{\mathsf{B}}(\mathbf{n}) \neq \emptyset$, and furthermore, for any $x \in F_{\mathsf{B}}(\mathbf{n})$ and any $y \in \mathcal{X} \setminus F_{\mathsf{B}}(\mathbf{n})$, we have $\mathbf{b}^{x,y}(\mathbf{n}) > 0$. This class of voting rules was introduced by Myerson (1995) and further explored by Pivato (2013), with the following result:

Lemma A.2. *Let $\mathcal{X}$ and $\mathcal{V}$ be arbitrary sets, and let $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ be a variable population anonymous voting rule. If F satisfies reinforcement, then F is a perfect balance rule.*

Proof (*Pivato, 2013*, *Lemma B2*). □

Given any linearly ordered Abelian group $\mathcal{R}$, an *$\mathcal{R}$-valued score system* on $(\mathcal{X}, \mathcal{V})$ is an $\mathcal{X}$-indexed collection $\mathsf{S} := \{\mathbf{s}^x\}_{x\in\mathcal{X}} \subset \mathcal{R}^{\mathcal{V}}$. The *scoring rule* $F_{\mathsf{S}} : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ again defined by formula (A.1). For any $\pi \in \Pi_{\mathcal{V}}$ and any $\mathbf{r} \in \mathcal{R}^{\mathcal{V}}$, we define $\mathbf{r}\pi \in \mathcal{R}^{\mathcal{V}}$ by setting $(\mathbf{r}\pi)_v := r_{\pi(v)}$ for all $v \in \mathcal{V}$. Let $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a group homomorphism. An $\mathcal{R}$-valued score system $\mathsf{S} = \{\mathbf{s}^x\}_{x\in\mathcal{X}}$ is *ν-neutral* if, for all $\pi \in \Pi_{\mathcal{X}}$ and $x, y \in \mathcal{X}$, if $\pi(y) = x$ and $\widetilde{\pi} := \nu(\pi)$, then $\mathbf{s}^x\,\widetilde{\pi} = \mathbf{s}^y$. More generally, S is *neutral* if it is ν-neutral for some group homomorphism $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$.

Proposition A.3. *Let $\mathcal{R}$ be a linearly ordered Abelian group. Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be any set, and let $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ be a balance rule determined by a perfect $\mathcal{R}$-valued balance system. Then F is neutral if and only if F is a scoring rule with an $\mathcal{R}$-valued neutral score system.*

Proof (*Pivato, 2013*, *Proposition B9*). □

A voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ is *nondegenerate* if, for all $x \in \mathcal{X}$, there is some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $F(\mathbf{n}) = \{x\}$.

[19] That is, $(\mathcal{R}, +)$ is a group, and the operation $+$ is *commutative*: $r + s = s + r$ for all $r, s \in \mathcal{R}$. Typically, the identity of an Abelian group is denoted by "0", and the inverse of the element $r \in \mathcal{R}$ is denoted by "$-r$" (Dummit and Foote, 2004). Again, we normally abuse notation and refer to "the linearly ordered Abelian group $\mathcal{R}$", when the operation $+$ and order $>$ are clear from context.

Lemma A.4. *Suppose $\mathcal{X}$ is finite. If a voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ is neutral, nontrivial, and satisfies reinforcement, then F is nondegenerate.*

Proof. Let $N := |\mathcal{X}|$ (finite).

Claim 1: Let $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, and let $M := |\mathcal{X} \setminus F(\mathbf{n})|$ (so $|F(\mathbf{n})| = N - M$).

(a) *If $|F(\mathbf{n})| \leq N/2$, then there exists $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n}')| = 1$.*
(b) *If $|F(\mathbf{n})| > N/2$, then there exists $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n}')| = N - 2M$.*

Proof. Let $\mathcal{Y} := F(\mathbf{n}) \subseteq \mathcal{X}$. If $|\mathcal{Y}| \leq N/2$, then there exists some $\mathcal{Y}' \subseteq \mathcal{X}$ such that $|\mathcal{Y}'| = |\mathcal{Y}|$ and $|\mathcal{Y} \cap \mathcal{Y}'| = 1$.

If $|\mathcal{Y}| > N/2$, then there exists some $\mathcal{Y}' \subseteq \mathcal{X}$ such that $|\mathcal{Y}'| = |\mathcal{Y}|$ and $\mathcal{X} \setminus \mathcal{Y}'$ is disjoint from $\mathcal{X} \setminus \mathcal{Y}$, and thus,

$$|\mathcal{X} \setminus (\mathcal{Y}' \cap \mathcal{Y})| = |(\mathcal{X} \setminus \mathcal{Y}') \sqcup (\mathcal{X} \setminus \mathcal{Y})| = |\mathcal{X} \setminus \mathcal{Y}'| + |\mathcal{X} \setminus \mathcal{Y}| = 2|\mathcal{X} \setminus \mathcal{Y}| = 2M.$$

Thus, $|\mathcal{Y}' \cap \mathcal{Y}| = |\mathcal{X}| - 2M = N - 2M$.

In either case, $|\mathcal{Y}'| = |\mathcal{Y}|$, so there exists $\pi \in \Pi_{\mathcal{X}}$ with $\pi(\mathcal{Y}) = \mathcal{Y}'$. Let $\widetilde{\pi} \in \Pi_{\mathcal{V}}$ be the image of π under the neutralizer homomorphism. Thus, $F[\widetilde{\pi}(\mathbf{n})] = \mathcal{Y}'$, by neutrality. Let $\mathbf{n}' := \mathbf{n} + \widetilde{\pi}(\mathbf{n})$. By construction $\mathcal{Y}' \cap \mathcal{Y} \neq \emptyset$; thus, $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ and $F(\mathbf{n}') = \mathcal{Y}' \cap \mathcal{Y}$, by reinforcement. ◊ Claim 1

Claim 2: There exists $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n}')| = 1$.

Proof. Since F is nontrivial, there exists some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n})| < N$. Now applying Claim 1(b) repeatedly, we can obtain some $\mathbf{n}'' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n}'')| \leq N/2$. Then apply Claim 1(a) to obtain some $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n}')| = 1$. ◊ Claim 2

Now, let $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ be from Claim 2. Thus, $F(\mathbf{n}') = \{x\}$ for some $x \in \mathcal{X}$. Let $y \in \mathcal{X}$. Find $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(x) = y$. Let $\widetilde{\pi} \in \Pi_{\mathcal{V}}$ be the image of π under the neutralizer homomorphism. Then $F[\widetilde{\pi}(\mathbf{n}')] = \pi[F(\mathbf{n})] = \{y\}$, by neutrality. This works for any $y \in \mathcal{X}$; thus, F is nondegenerate. □

Lemma A.5. *For any $\mathbf{b} \in \mathcal{R}^{\mathcal{V}}$, $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ and $\pi \in \Pi_{\mathcal{V}}$, we have $(\mathbf{b}\pi)(\mathbf{n}) = \mathbf{b}(\pi(\mathbf{n}))$.*

Proof.

$$(\mathbf{b}\pi)(\mathbf{n}) = \sum_{v\in\mathcal{V}} n_v(\mathbf{b}\pi)_v = \sum_{v\in\mathcal{V}} n_v b_{\pi(v)} \underset{(*)}{=} \sum_{v'\in\mathcal{V}} n_{\pi^{-1}(v')} b_{v'} = \sum_{v\in\mathcal{V}} \pi(\mathbf{n})_{v'} b_{v'} = \mathbf{b}(\pi(\mathbf{n})).$$

Here, $(*)$ is the change of variables $v' := \pi(v)$. □

A voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies the *tie condition* (TC) if, for all distinct $x, y \in \mathcal{X}$:

(TC1) There exists some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $F(\mathbf{n}) = \{x, y\}$.
(TC2) For any finite $\mathcal{W} \subseteq \mathcal{V}$, there exists some $\mathbf{m} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ such that $m_w > 0$ for all $w \in \mathcal{W}$, and $F(\mathbf{m}) \supseteq \{x, y\}$.

Lemma A.6. *Suppose $\mathcal{X}$ is finite, and let $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ be a voting rule.*

(a) *If F is neutral, then F satisfies* (TC2) *of the tie condition.*
(b) *If F is neutral, nontrivial, and satisfies reinforcement, then F satisfies* (TC1).

Proof. (a) Fix $x, y \in \mathcal{X}$. Let $\mathcal{W} \subseteq \mathcal{V}$ be finite, and define $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ by $n_w := 1$ for all $w \in \mathcal{W}$, while $n_v := 0$ for all $v \in \mathcal{V} \setminus \mathcal{W}$. Define $\overline{\mathbf{n}} := \sum_{\pi\in\Pi_{\mathcal{X}}} \widetilde{\pi}(\mathbf{n})$ (here, for all $\pi \in \Pi_{\mathcal{X}}$, we use $\widetilde{\pi} \in \Pi_{\mathcal{V}}$ to denote the image of π under the neutralizer homomorphism). Then $\overline{\mathbf{n}}$ is $\widetilde{\Pi}_{\mathcal{X}}$-fixed, so neutrality implies that $F(\overline{\mathbf{n}})$ is a $\Pi_{\mathcal{X}}$-invariant subset of $\mathcal{X}$. Since $F(\overline{\mathbf{n}}) \neq \emptyset$, this means that $F(\overline{\mathbf{n}}) = \mathcal{X}$. In particular, $\{x, y\} \subseteq F(\overline{\mathbf{n}})$. Finally, for all $w \in \mathcal{W}$, we have $\overline{n}_w \geq n_w = 1 \geq 0$, as required by (TC2).

(b) If $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies reinforcement, then Lemma A.2 says F is a perfect balance rule. If F is also neutral, then Proposition A.3 says that F is a scoring rule with a neutral scoring system S. Meanwhile, Lemma A.4 yields some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F(\mathbf{n})| = 1$. Let $F(\mathbf{n}) = \{x\}$ for some $x \in \mathcal{X}$. There are now two cases: either $|\mathcal{X}| = 2$, or $|\mathcal{X}| \geq 3$.

- If $|\mathcal{X}| = 2$, then let y be the only other element in $\mathcal{X}$ besides x, and let $\pi \in \Pi_{\mathcal{X}}$ be the permutation such that $\pi(x) = y$ and $\pi(y) = x$.
- If $|\mathcal{X}| \geq 3$, then, since $\mathcal{X}$ is finite, there exists $y \in \mathcal{X}$ such that

$$\mathbf{s}^x(\mathbf{n}) > \mathbf{s}^y(\mathbf{n}) \geq \mathbf{s}^z(\mathbf{n}), \quad \text{for all } z \in \mathcal{X} \setminus \{x, y\}. \tag{A.2}$$

In this case, let $\pi \in \Pi_{\mathcal{X}}$ be the permutation such that $\pi(x) = y$ and $\pi(y) = x$, while $\pi(z) = z$ for all other $z \in \mathcal{X} \setminus \{x, y\}$.

In either case, $\pi^2 = \mathrm{Id}$. Let $\widetilde{\pi} \in \Pi_{\mathcal{V}}$ be the image of π under the neutralizer homomorphism. Define $\overline{\mathbf{n}} := \mathbf{n} + \widetilde{\pi}(\mathbf{n})$.

Claim 1: $F(\overline{\mathbf{n}}) = \{x, y\}$.

Proof. Note that

$$\mathbf{s}^x(\widetilde{\pi}(\mathbf{n})) \underset{(*)}{=} (\mathbf{s}^x\widetilde{\pi})(\mathbf{n}) \underset{(\dagger)}{=} \mathbf{s}^{\pi^{-1}(x)}(\mathbf{n}) = \mathbf{s}^y(\mathbf{n}), \tag{A.3}$$

$$\text{and} \quad \mathbf{s}^y(\widetilde{\pi}(\mathbf{n})) \underset{(*)}{=} (\mathbf{s}^y\widetilde{\pi})(\mathbf{n}) \underset{(\dagger)}{=} \mathbf{s}^{\pi^{-1}(y)}(\mathbf{n}) = \mathbf{s}^x(\mathbf{n}). \tag{A.4}$$

Here, both $(*)$ are by Lemma A.5, and both $(\dagger)$ are by neutrality. Thus,

$$\begin{aligned} \mathbf{s}^x(\overline{\mathbf{n}}) &= \mathbf{s}^x(\mathbf{n} + \widetilde{\pi}(\mathbf{n})) = \mathbf{s}^x(\mathbf{n}) + \mathbf{s}^x(\widetilde{\pi}(\mathbf{n})) \\ &\underset{(*)}{=} \mathbf{s}^x(\mathbf{n}) + \mathbf{s}^y(\mathbf{n}), \end{aligned} \tag{A.5}$$

$$\begin{aligned} \text{and} \quad \mathbf{s}^y(\overline{\mathbf{n}}) &= \mathbf{s}^y(\mathbf{n} + \widetilde{\pi}(\mathbf{n})) = \mathbf{s}^y(\mathbf{n}) + \mathbf{s}^y(\widetilde{\pi}(\mathbf{n})) \\ &\underset{(\dagger)}{=} \mathbf{s}^y(\mathbf{n}) + \mathbf{s}^x(\mathbf{n}). \end{aligned} \tag{A.6}$$

Here, $(*)$ is by Eq. (A.3), and $(\dagger)$ is by Eq. (A.4). Eqs. (A.5) and (A.6) imply that $\mathbf{s}^x(\overline{\mathbf{n}}) = \mathbf{s}^y(\overline{\mathbf{n}})$. If $\mathcal{X} = \{x, y\}$, then this implies that $F(\overline{\mathbf{n}}) = \{x, y\}$.

On the other hand, if $|\mathcal{X}| \geq 3$, then for any $z \in \mathcal{X} \setminus \{x, y\}$, we also have

$$\mathbf{s}^z(\overline{\mathbf{n}}) = \mathbf{s}^z(\mathbf{n} + \widetilde{\pi}(\mathbf{n})) = \mathbf{s}^z(\mathbf{n}) + \mathbf{s}^z(\widetilde{\pi}(\mathbf{n})), \tag{A.7}$$

$$\text{and} \quad \mathbf{s}^z(\widetilde{\pi}(\mathbf{n})) \underset{(*)}{=} (\mathbf{s}^z\widetilde{\pi})(\mathbf{n}) \underset{(\dagger)}{=} \mathbf{s}^{\pi^{-1}(z)}(\mathbf{n}) \underset{(\diamond)}{\leq} \mathbf{s}^y(\mathbf{n}) \underset{(\diamond)}{<} \mathbf{s}^x(\mathbf{n}). \tag{A.8}$$

Here, $(*)$ is by Lemma A.5, $(\dagger)$ is by neutrality, and both $(\diamond)$ are by inequality (A.2). Combining Eqs. (A.5)–(A.7) with inequalities (A.2) and (A.8), we conclude that $\mathbf{s}^z(\overline{\mathbf{n}}) < \mathbf{s}^x(\overline{\mathbf{n}}) = \mathbf{s}^y(\overline{\mathbf{n}})$, for all $z \in \mathcal{X} \setminus \{x, y\}$. Thus, $F(\overline{\mathbf{n}}) = \{x, y\}$, as desired. ◊ Claim 1

Now, let $x', y' \in \mathcal{X}$. Find $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(x) = x'$ and $\pi(y) = y'$. Let $\widetilde{\pi} \in \Pi_{\mathcal{V}}$ be the image of π under the neutralizer homomorphism. Then

$$F[\widetilde{\pi}(\overline{\mathbf{n}})] \underset{(N)}{=} \pi[F(\overline{\mathbf{n}})] \underset{(*)}{=} \pi\{x, y\} = \{x', y'\},$$

as desired. Here, (N) is by neutrality, and $(*)$ is by Claim 1. □

Let $(\mathcal{R}, +, >)$ be a linearly ordered Abelian group, and let $r, s \in \mathcal{R}$ be positive. We say r is *infinitesimal* relative to s if $N\,r < s$ for all $N \in \mathbb{N}$. We say $(\mathcal{R}, +, >)$ is *Archimedean* if it has no infinitesimal elements. For example $\mathbb{R}$ (with the standard ordering) is Archimedean, but $\mathbb{R}^N_{\mathrm{lex}}$ is not, if $N \geq 2$. We state the next result for future reference.

Hölder's theorem. *$(\mathcal{R}, +, >)$ is Archimedean if and only if it is isomorphic to a subgroup of $(\mathbb{R}, +, >)$.*

Proof. See Theorem 1 on p. 45, Chapter IV.1 of Fuchs (2011). □

Let $\mathbb{Z}^{\langle\mathcal{V}\rangle} := \{\mathbf{n} \in \mathbb{Z}^{\mathcal{V}};\ \|\mathbf{n}\| < \infty\}$; then $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ is an Abelian group. If $\mathcal{R}$ is another Abelian group, then there is a bijective correspondence between $\mathcal{R}^{\mathcal{V}}$ and the set of group homomorphisms from $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ into $\mathcal{R}$. To see this, note that any vector $\mathbf{b} = (b_v)_{v\in\mathcal{V}} \in \mathcal{R}^{\mathcal{V}}$ defines a group homomorphism $\mathbf{b}^* : \mathbb{Z}^{\langle\mathcal{V}\rangle} \longrightarrow \mathcal{R}$ by setting $\mathbf{b}^*(\mathbf{z}) := \sum_{v\in\mathcal{V}} z_v b_v$ for all $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$. Conversely, for all $v \in \mathcal{V}$, let $\mathbf{1}_v \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$ be the vector with a 1 in the v coordinate, and 0 in all other coordinates. Then given any group homomorphism $\beta : \mathbb{Z}^{\langle\mathcal{V}\rangle} \longrightarrow \mathcal{R}$, we can define a vector $\mathbf{b} = (b_v)_{v\in\mathcal{V}} \in \mathcal{R}^{\mathcal{V}}$ by setting $b_v := \beta(\mathbf{1}_v)$ for all $v \in \mathcal{V}$. We then have $\mathbf{b}^* = \beta$.

Thus, we can equivalently define a balance system $\{\mathbf{b}^{x,y}\}_{x,y\in\mathcal{X}}$ as a collection of $\mathcal{R}$-valued *group homomorphisms* on $\mathbb{Z}^{\langle\mathcal{V}\rangle}$, rather than as a collection of vectors in $\mathcal{R}^{\mathcal{V}}$. This convention should be kept in mind when reading the next proof.

The next result is a key step in the proof of Proposition A.1. It shows how the axiom of overwhelming majority causes a balance rule (or scoring rule) to be real-valued, by forcing the underlying group $\mathcal{R}$ to be Archimedean.

Proposition A.7. *Let $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ be a perfect balance rule satisfying* TC. *Then F satisfies overwhelming majority if and only if $F = F_{\mathsf{B}}$ for some $\mathbb{R}$-valued balance system B.*

Proof. "$\Longleftarrow$" is straightforward.

"$\Longrightarrow$" By hypothesis, $F = F_{\widetilde{\mathsf{B}}}$ for some balance system $\widetilde{\mathsf{B}} := \{\widetilde{\mathbf{b}}^{x,y}\}_{x,y\in\mathcal{X}}$ taking values in some linearly ordered Abelian group $\widetilde{\mathcal{R}}$. Fix $x, y \in \mathcal{X}$. Let $\mathcal{R}'_{x,y} := \widetilde{\mathbf{b}}^{x,y}(\mathbb{Z}^{\langle\mathcal{V}\rangle}) \subseteq \widetilde{\mathcal{R}}$. Then $\mathcal{R}'_{x,y}$ is also a linearly ordered Abelian group, and we can treat $\widetilde{\mathbf{b}}^{x,y}$ as a group homomorphism from $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ into $\mathcal{R}'_{x,y}$.

Claim 1: $\mathcal{R}'_{x,y} = \widetilde{\mathbf{b}}^{x,y}(\mathbb{N}^{\langle\mathcal{V}\rangle})$.

Proof. Let $r \in \mathcal{R}'_{x,y}$; then $r = \widetilde{\mathbf{b}}^{x,y}(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$. Let $\mathcal{W} := \{v \in \mathcal{V};\ z_v \neq 0\}$ (a finite set). Condition (TC2) yields some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $n_w > 0$ for all $w \in \mathcal{W}$, such that $F(\mathbf{n}) \supseteq \{x, y\}$. Thus, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}) = 0$, so $\mathbf{n} \in \ker(\widetilde{\mathbf{b}}^{x,y})$. Thus, $M\mathbf{n} \in \ker(\widetilde{\mathbf{b}}^{x,y})$ for all $M \in \mathbb{N}$.

Let $M = 1 + \max\{|z_w|/n_w;\ w \in \mathcal{W}\}$ (so M is finite, because $|\mathcal{W}| < \infty$). Thus, $Mn_w + z_w > 0$ for all $w \in \mathcal{W}$. Thus, $M\mathbf{n} + \mathbf{z} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, and clearly, $\widetilde{\mathbf{b}}^{x,y}(M\mathbf{n} + \mathbf{z}) = M \cdot \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}) + \widetilde{\mathbf{b}}^{x,y}(\mathbf{z}) = M \cdot 0 + r = r$, as desired. ◊ Claim 1

Claim 2: *$\mathcal{R}'_{x,y}$ is Archimedean for all $x, y \in \mathcal{X}$.*

Proof. Let $r_1, r_2 \in \mathcal{R}_{x,y}$, with $r_1 > 0$. We must find some $N \in \mathbb{N}$ such that $N \cdot r_1 > -r_2$. By Claim 1, there exist $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ such that $r_1 = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_1)$ and $r_2 = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_2)$. Condition (TC1) yields some $\mathbf{n}_0 \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ such that $F(\mathbf{n}_0) = \{x, y\}$. By overwhelming majority, there exists some $M \in \mathbb{N}$ such that $F(\mathbf{n}_1 + M\mathbf{n}_0) \subseteq \{x, y\}$. Let $\mathbf{n}'_1 := \mathbf{n}_1 + M\mathbf{n}_0$. Then $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_1) = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_1) + M \cdot \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_0) = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_1) = r_1$, because $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_0) = 0$ because $F(\mathbf{n}_0) = \{x, y\}$. Thus, $F(\mathbf{n}'_1) = \{x\}$, because $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_1) = r_1 > 0$ and $F(\mathbf{n}'_1) \subseteq \{x, y\}$ by construction. By overwhelming majority, there exists some $N \in \mathbb{N}$ such that $F(N\,\mathbf{n}'_1 + \mathbf{n}_2) = \{x\}$. But this means that $0 < \widetilde{\mathbf{b}}^{x,y}(N\,\mathbf{n}'_1 + \mathbf{n}_2) = N\,\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_1) + \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_2) = N\,r_1 + r_2$. Thus, $N\,r_1 > -r_2$, as desired. ◊ Claim 2

For all $x, y \in \mathcal{X}$, Hölder's theorem and Claim 2 imply that $\mathcal{R}'_{x,y}$ is isomorphic to some ordered subgroup of $\mathbb{R}$; thus, we can regard $\widetilde{\mathbf{b}}^{x,y}$ as a real-valued function, so that $\widetilde{\mathsf{B}}$ is a real-valued balance system. □

Proof of Proposition A.1. If $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies reinforcement, then Lemma A.2 says F is a perfect balance rule. If F is also neutral and nontrivial, then Lemma A.6 says that F satisfies the tie condition. Thus, if F also satisfies overwhelming majority, then Proposition A.7 says F is a real-valued perfect balance rule. Since F is neutral, Proposition A.3 then says that $F = F_{\mathsf{S}}$, where S is a neutral, real-valued score system. □

Proof of Theorem 1. Let $\mathcal{F}$ be the class of neutral, $\mathcal{X}$-valued voting rules which satisfy reinforcement and overwhelming majority; we seek the most expressive member of $\mathcal{F}$ (if it exists). Let G be any nontrivial, neutral, $\mathcal{X}$-valued scoring rule with a real-valued score system (e.g. approval voting). Then G is in $\mathcal{F}$. Thus, if F is the *most* expressive member of $\mathcal{F}$, then F must be at least as expressive as G, which means F itself must be nontrivial. However, Proposition A.1 says that any nontrivial rule in $\mathcal{F}$ must be a scoring rule with a neutral, real-valued score system. Thus, it suffices to confine our attention to such scoring rules.

So, consider a scoring rule F_{S}, where S is a neutral, real-valued score system on $(\mathcal{X}, \mathcal{V})$ (for some signal set $\mathcal{V}$). For any $v \in \mathcal{V}$, let $\mathbf{s}_v := (s_v^x)_{x\in\mathcal{X}}$, a vector in $\mathbb{R}^{\mathcal{X}}$. Let $\mathsf{S}^\dagger := \{\mathbf{s}_v\}_{v\in\mathcal{V}}$; then $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$.

Claim 1: Suppose F_{S} is the most expressive scoring rule with $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$. Then either $\mathsf{S}^\dagger = \mathbb{R}^{\mathcal{X}}$, or F_{S} is equivalent to a scoring rule $F_{\widetilde{\mathsf{S}}}$ with a score system $\widetilde{\mathsf{S}}$ such that $\widetilde{\mathsf{S}}^\dagger = \mathbb{R}^{\mathcal{X}}$.

Proof. If $\mathsf{S}^\dagger = \mathbb{R}^{\mathcal{X}}$, then we are done. So, suppose that $\mathsf{S}^\dagger \subsetneq \mathbb{R}^{\mathcal{X}}$. Let $\mathcal{U} := \mathbb{R}^{\mathcal{X}} \setminus \mathsf{S}^\dagger$. Assume $\mathcal{U}$ is disjoint from $\mathcal{V}$. (This is without loss of generality, because the elements of $\mathcal{V}$ are just abstract "signals"; if some of these signals happen to be elements of $\mathcal{U}$, then we can just replace them with other signals which are not elements of $\mathcal{U}$.) Define $\mathcal{W} := \mathcal{V}\sqcup\mathcal{U}$. Define the score system $\widetilde{\mathsf{S}} \subset \mathbb{R}^{\mathcal{W}}$ by setting $\widetilde{s}_v^x := s_v^x$ for all $v \in \mathcal{V}$ and $x \in \mathcal{X}$, whereas $\widetilde{s}_{\mathbf{u}}^x := u^x$ for all $\mathbf{u} \in \mathcal{U}$ and $x \in \mathcal{X}$. In other words, $\widetilde{\mathbf{s}}_v := \mathbf{s}_v$ for all $v \in \mathcal{V}$, while $\widetilde{\mathbf{s}}_{\mathbf{u}} := \mathbf{u}$ for all $\mathbf{u} \in \mathcal{U}$. Thus, $\widetilde{\mathsf{S}}^\dagger = \mathbb{R}^{\mathcal{X}}$, by construction.

Now, let $\alpha : \mathcal{V} \longrightarrow \mathcal{W}$ be the inclusion map, and let $\alpha_* : \mathbb{N}^{\langle\mathcal{V}\rangle} \longrightarrow \mathbb{N}^{\langle\mathcal{W}\rangle}$ be defined as prior to Theorem 1. Then for any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, and any $x \in \mathcal{X}$ it is clear that $\widetilde{\mathbf{s}}^x[\alpha_*(\mathbf{n})] = \mathbf{s}^x(\mathbf{n})$. Thus, $F_{\widetilde{\mathsf{S}}}[\alpha_*(\mathbf{n})] = F_{\mathsf{S}}(\mathbf{n})$. Thus, $F_{\widetilde{\mathsf{S}}}$ is at least as expressive as F_{S}. But by the hypothesis, F_{S} is the most expressive rule with $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$. Thus, we conclude that $F_{\widetilde{\mathsf{S}}}$ must actually be equivalent to F_{S}. ◊ Claim 1

Now, suppose F_{S} is the most expressive scoring rule with $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$. We will show that F_{S} is equivalent to the formal utilitarian voting rule. Claim 1 implies that we can suppose $\mathsf{S}^\dagger = \mathbb{R}^{\mathcal{X}}$ without loss of generality (because we only care about the equivalence class of F_{S}). For any $v, w \in \mathcal{V}$, if $\mathbf{s}_v = \mathbf{s}_w$, then a vote for v has the same effect as a vote for w, when added to any profile. Thus, we can regard v and w as the same. Thus, for each $\mathbf{r} \in \mathbb{R}^{\mathcal{X}}$, there exists a unique $v \in \mathcal{V}$ with $\mathbf{s}_v = \mathbf{r}$. At this point it is clear that F_{S} is the formal utilitarian voting rule. □

The proof of Theorem 2 also requires the next lemma.

Lemma A.8. *Let $\mathcal{X}$ be a finite set, and let S be a real-valued scoring system on $\mathcal{X}$.*

(a) *Suppose there exists $R \in \mathbb{R}_+$ such that $|s_v^x - s_v^y| \le R$ for all $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$. Then F_{S} does not admit minority overrides.*
(b) *Suppose S is neutral and nontrivial. If F_{S} does not admit minority overrides, then there is some $R \in \mathbb{R}$ such that $|s_v^x - s_v^y| \le R$ for all $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$.*

Proof. (a) Let $M := \min\{|F_{\mathsf{S}}(\mathbf{n})|;\ \mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}\}$ (so $M \ge 1$). Find $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F_{\mathsf{S}}(\mathbf{n})| = M$. Fix $x \in F_{\mathsf{S}}(\mathbf{n})$, and let $\delta := \min\{\mathbf{s}^x(\mathbf{n}) - \mathbf{s}^y(\mathbf{n});\ y \in \mathcal{X}\setminus F_{\mathsf{S}}(\mathbf{n})\}$; then $\delta > 0$ and is well-defined because $\mathcal{X}$ is finite. Let $K := \lceil R/\delta\rceil + 1$, and let $\mathbf{n}' := K\,\mathbf{n}$; then $F_{\mathsf{S}}(\mathbf{n}') = F_{\mathsf{S}}(\mathbf{n})$, by reinforcement.

Claim 1: For any $v \in \mathcal{V}$, we have $F_{\mathsf{S}}(\mathbf{n}' + \mathbf{1}^v) = F_{\mathsf{S}}(\mathbf{n})$.

Proof. For any $y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})$, we have

$$\begin{aligned}\mathbf{s}^y(\mathbf{n}' + \mathbf{1}^v) &= K\mathbf{s}^y(\mathbf{n}) + \mathbf{s}^y(\mathbf{1}^v) = K\mathbf{s}^y(\mathbf{n}) + s_v^y\\ &\underset{(\dagger)}{\le} K\mathbf{s}^x(\mathbf{n}) - K\delta + s_v^x + R \underset{(*)}{<} K\mathbf{s}^x(\mathbf{n}) + s_v^x\\ &= K\mathbf{s}^x(\mathbf{n}) + \mathbf{s}^x(\mathbf{1}^v) = \mathbf{s}^x(\mathbf{n}' + \mathbf{1}^v).\end{aligned}$$

(Here, $(\dagger)$ is because $\mathbf{s}^y(\mathbf{n}) \le \mathbf{s}^x(\mathbf{n}) - \delta$ (because $y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})$), while $s_v^y \le s_v^x + R$, by definition of R. Next, $(*)$ is because $K\delta > R$, by definition of K.) Thus, $y \notin F_{\mathsf{S}}(\mathbf{n}' + \mathbf{1}^v)$. This holds for all $y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})$, so we conclude that $F_{\mathsf{S}}(\mathbf{n}' + \mathbf{1}^v) \subseteq F_{\mathsf{S}}(\mathbf{n}')$. But then $F_{\mathsf{S}}(\mathbf{n}' + \mathbf{1}^v) = F_{\mathsf{S}}(\mathbf{n}')$, because $|F_{\mathsf{S}}(\mathbf{n}')| = M$ is already of minimal size. ◊ Claim 1

Claim 1 shows that F does not admit minority overrides.

(b) (by contrapositive) Let $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be the neutralizer of S. Suppose that the conclusion is false; suppose that, for all $R \in \mathbb{R}$, there exists some $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$ with $|s_v^x - s_v^y| > R$. We will show that F admits minority overrides.

Claim 2: For all $R \in \mathbb{R}$ and all $x \in \mathcal{X}$, there exist $v \in \mathcal{V}$ and $y \in \mathcal{X}$ with $s_v^y - s_v^x > R$.

Proof. Given $R \in \mathbb{R}$, there exists some $w \in \mathcal{V}$ and $a, b \in \mathcal{X}$ with $|s_w^a - s_w^b| > R$. First suppose $s_w^b > s_w^a$, so that in fact $s_w^b - s_w^a > R$. Find $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(x) = a$. Let $y := \pi^{-1}(b)$, let $\widetilde{\pi} := \nu(\pi)$ and let $v := \widetilde{\pi}^{-1}(w)$. Since S is ν-neutral, we have $\mathbf{s}^x = (\mathbf{s}^a\widetilde{\pi})$; thus $s_v^x = (\mathbf{s}^a\widetilde{\pi})_v = s_{\widetilde{\pi}(v)}^a = s_w^a$, and likewise, $s_v^y = s_w^b$; thus $s_v^y - s_v^x > R$.

Now suppose $s_w^b < s_w^a$, so that $s_w^a - s_w^b > R$. In this case, find $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(x) = b$. Let $y := \pi^{-1}(a)$, let $\widetilde{\pi} := \nu(\pi)$ and let $v := \widetilde{\pi}^{-1}(w)$. Then by a similar argument we have $s_v^y = s_w^a$ and $s_v^x = s_w^b$; thus $s_v^y - s_v^x > R$. ◊ Claim 2

Let $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ be any profile with $F_{\mathsf{S}}(\mathbf{n}) \neq \mathcal{X}$. Fix $x \in F_{\mathsf{S}}(\mathbf{n})$. Let $R := \max\{\mathbf{s}^x(\mathbf{n}) - \mathbf{s}^y(\mathbf{n});\ y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})\}$; then $R > 0$ and is well-defined because $\mathcal{X}$ is finite. Claim 2 yields some $v \in \mathcal{V}$ and $y \in \mathcal{X}$ with $s_v^y - s_v^x > R$. There are now two cases: either $y \in F_{\mathsf{S}}(\mathbf{n})$, or $y \notin F_{\mathsf{S}}(\mathbf{n})$.

Case 1. If $y \in F_{\mathsf{S}}(\mathbf{n})$, then

$$\begin{aligned}\mathbf{s}^y(\mathbf{n} + \mathbf{1}^v) &= \mathbf{s}^y(\mathbf{n}) + \mathbf{s}^y(\mathbf{1}^v) = \mathbf{s}^y(\mathbf{n}) + s_v^y \underset{(*)}{=} \mathbf{s}^x(\mathbf{n}) + s_v^y\\ &\underset{(\dagger)}{>} \mathbf{s}^x(\mathbf{n}) + s_v^x = \mathbf{s}^x(\mathbf{n} + \mathbf{1}^v).\end{aligned}$$

Thus, $x \notin F_{\mathsf{S}}(\mathbf{n} + \mathbf{1}^v)$, so $F_{\mathsf{S}}(\mathbf{n} + \mathbf{1}^v) \neq F_{\mathsf{S}}(\mathbf{n})$. (Here $(*)$ is because $\mathbf{s}^y(\mathbf{n}) = \mathbf{s}^x(\mathbf{n})$ because $\{x, y\} \in F_{\mathsf{S}}(\mathbf{n})$. Meanwhile, $(\dagger)$ is because $s_v^y > s_v^x + R > s_v^x$.)

Case 2. If $y \notin F_{\mathsf{S}}(\mathbf{n})$, then

$$\begin{aligned}\mathbf{s}^y(\mathbf{n} + \mathbf{1}^v) &= \mathbf{s}^y(\mathbf{n}) + s_v^y \ge \mathbf{s}^x(\mathbf{n}) - R + s_v^y\\ &\underset{(\dagger)}{>} \mathbf{s}^x(\mathbf{n}) - R + R + s_v^x = \mathbf{s}^x(\mathbf{n}) + s_v^x = \mathbf{s}^x(\mathbf{n} + \mathbf{1}^v).\end{aligned}$$

Thus, again $x \notin F_{\mathsf{S}}(\mathbf{n} + \mathbf{1}^v)$, so $F_{\mathsf{S}}(\mathbf{n} + \mathbf{1}^v) \neq F_{\mathsf{S}}(\mathbf{n})$. (Here, $(\dagger)$ is because $s_v^y > s_v^x + R$.)

This construction works for any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$; thus, F_{S} admits minority overrides. □

Proof of Theorem 2. Let $\mathcal{F}_0$ be the class of neutral, $\mathcal{X}$-valued voting rules which satisfy reinforcement, overwhelming majority, and admit no minority overrides. We seek the most expressive member of $\mathcal{F}_0$ (if it exists). Let G be any nontrivial, neutral, $\mathcal{X}$-valued scoring rule with a score system ranging over $[0, 1]$ (e.g. approval voting). Then G is in $\mathcal{F}_0$. Thus, if F is the *most* expressive member of $\mathcal{F}_0$, then F must be at least as expressive as G, which means F itself must be nontrivial. However, Proposition A.1 says that any nontrivial rule in $\mathcal{F}_0$ must be a scoring rule with a neutral, real-valued score system. Thus, it again suffices to confine our attention to such scoring rules. So, consider a scoring rule F_{S}, where S is a neutral, real-valued score system on $(\mathcal{X}, \mathcal{V})$ (for some signal set $\mathcal{V}$).

Claim 1: If F_{S} does not admit minority overrides, then there is a real-valued scoring system $\widetilde{\mathsf{S}} = \{\widetilde{\mathbf{s}}^x\}_{x\in\mathcal{X}}$ with $0 \le \widetilde{s}_v^x \le 1$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$, such that $F_{\widetilde{\mathsf{S}}} = F_{\mathsf{S}}$.

Proof. Lemma A.8 says that F does not admit minority overrides if and only if there is some $R \in \mathbb{R}_+$ such that $|s_v^x - s_v^y| \le R$ for all $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$. Let $r := 1/R$; thus, $|r\,s_v^x - r\,s_v^y| \le 1$ for all $x, y \in \mathcal{X}$ and $v \in \mathcal{V}$. Now, for each $v \in \mathcal{V}$, let $t_v := \min\{r\,s_v^x;\ x \in \mathcal{X}\}$, to obtain a vector $\mathbf{t} := (t_v)_{v\in\mathcal{V}}$ (this is well-defined because $\mathcal{X}$ is finite). Now define $\widetilde{\mathbf{s}}^x := r\,\mathbf{s}^x - \mathbf{t}$, for all $x \in \mathcal{X}$; then $\widetilde{\mathsf{S}}$ is an affine transform of S, so $F_{\widetilde{\mathsf{S}}} = F_{\mathsf{S}}$.

Now, for any $v \in \mathcal{V}$, we have $\min\{\widetilde{s}^x_v;\ x \in \mathcal{X}\} = 0$, by construction. Also, $\max\{\widetilde{s}^x_v - \widetilde{s}^y_v;\ x, y \in \mathcal{X}\} \leq 1$, which implies that $\max\{\widetilde{s}^x_v;\ x \in \mathcal{X}\} \leq 1$. Thus, $0 \leq \widetilde{s}^x_v \leq 1$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$. ◊ Claim 1

By replacing S by $\widetilde{\mathsf{S}}$ from Claim 1 if necessary, we can assume without loss of generality that $\widetilde{s}^x_v \in [0, 1]$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$. For any $v \in \mathcal{V}$, define $\mathbf{s}_v := (s^x_v)$, a vector in $[0, 1]^{\mathcal{X}}$. Let $\mathsf{S}^\dagger := \{\mathbf{s}_v\}_{v\in\mathcal{V}}$; then $\mathsf{S}^\dagger \subseteq [0, 1]^{\mathcal{X}}$.

At this point, the argument is very similar to the proof of Theorem 1. By an argument identical to Claim 1 in that proof, one can show: *If F_S is the most expressive rule with $\mathsf{S}^\dagger \subseteq [0, 1]^{\mathcal{X}}$, then* $\mathsf{S}^\dagger = [0, 1]^{\mathcal{X}}$. So, suppose $\mathsf{S}^\dagger = [0, 1]^{\mathcal{X}}$. For any $v, w \in \mathcal{V}$, if $\mathbf{s}_v = \mathbf{s}_w$, then a vote for v has the same effect as a vote for w, when added to any profile. Thus, we can regard v and w as the same. Thus, for each $\mathbf{t} \in [0, 1]^{\mathcal{X}}$, there exists a unique $v \in \mathcal{V}$ with $\mathbf{s}_v = \mathbf{t}$. At this point it is clear that F_S is the range voting rule. □

Proof of Proposition 3. It suffices to prove the theorem for range voting, since it is less expressive than formal utilitarian voting. Let $\mathcal{X}$ be a set with at least two elements, let $\mathcal{V} := [0, 1]^{\mathcal{X}}$ be the signal space for range voting, and let $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ be the range voting rule. Now, by contradiction, suppose that $\mathcal{W}$ is a countable signal set and $G : \mathbb{N}^{\langle\mathcal{W}\rangle} \rightrightarrows \mathcal{X}$ is another voting rule which is as expressive as range voting. That is, there exists some translation map $\alpha : \mathcal{V} \longrightarrow \mathcal{W}$ such that, for any range voting profile $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, we have $G[\alpha_*(\mathbf{n})] = F(\mathbf{n})$. We will derive a contradiction.

Let $x \in \mathcal{X}$, and for all $t \in [0, 1]$, let $v^x_t \in \mathcal{V}$ be a vote which assigns score t to x and zero to every other alternative. Thus, the set $\{v^x_t;\ t \in [0, 1]\} \subset \mathcal{V}$ is uncountable. Since $\mathcal{W}$ is countable, there must exist distinct $s, t \in [0, 1]$ such that $\alpha(v^x_s) = \alpha(v^x_t)$. Without loss of generality, suppose $s < t$.

Claim 1: *There exist some $N, M \in \mathbb{N}$ such that $N\,s < M < N\,t$.*

Proof. Let r be a rational number such that $0 < r < t - s$. Thus, if $q = 2/r$, then q is also rational, and $q\,(t - s) > 2$, which means there is some $k \in \mathbb{N}$ such that $q\,s < k < q\,t$. Suppose $q = N/m$ for some $N, m \in \mathbb{N}$. Then we get $N\,s < m\,k < N\,t$. Now let $M := m\,k$. ◊ Claim 1

Let $y \in \mathcal{X} \setminus \{x\}$, and let $v^y_1 \in \mathcal{V}$ be the vote which assigns score 1 to y and zero to all other alternatives. Let $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ be a range-voting profile consisting of $N+M$ voters, where N voters say v^x_s and M voters say v^y_1. Then x gets a score of $N\,s$ and y gets a score of $M > N\,s$, and all other alternatives get a score of zero, so $F(\mathbf{n}) = \{y\}$. Let $\mathbf{n}' \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ be a range-voting profile consisting of $N + M$ voters, where N voters say v^x_t and M voters say v^y_1. Then x gets a score of $N\,t$ and y gets a score of $M < N\,t$, and all other alternatives get a score of zero, so $F(\mathbf{n}') = \{x\}$. However, $\alpha_*(\mathbf{n}) = \alpha_*(\mathbf{n}')$, because $\alpha(v^x_s) = \alpha(v^x_t)$. Thus, $G[\alpha_*(\mathbf{n})] = G[\alpha_*(\mathbf{n}')]$, which implies $x = y$. Contradiction. □

Appendix B. Independence of the axioms

We will now show, via examples, that the axioms appearing in Theorems 1 and 2 are logically independent.

Neutrality. Let S be a neutral, real-valued scoring rule, with all scores ranging over $[0, 1]$. Fix some alternative $y \in \mathcal{X}$, and define a new real-valued scoring rule $\widetilde{\mathsf{S}}$ such that $\widetilde{\mathbf{s}}^x := \mathbf{s}^x$ for all $x \in \mathcal{X} \setminus \{y\}$, whereas $\widetilde{\mathbf{s}}^y := 2 \cdot \mathbf{s}^y$. Then $F_{\widetilde{\mathsf{S}}}$ is an anonymous, variable-population rule which satisfies reinforcement and overwhelming majority, and does not admit minority overrides. But $F_{\widetilde{\mathsf{S}}}$ is *not* neutral, because y is treated differently from the other alternatives in $\mathcal{X}$.

Overwhelming majority. Let F_{S_1} and F_{S_2} be two different neutral, $[0, 1]$-valued scoring rules (e.g. the Borda rule and the approval voting rule). Define a compound voting rule F as follows. First, apply F_{S_1}; if this rule yields a unique winner, then stop. Otherwise, apply F_{S_2} only to break the tie between the winners of F_{S_1}. Then F is a neutral, anonymous, variable-population rule which satisfies reinforcement and does not admit minority overrides. But F does *not* satisfy overwhelming majority.

Reinforcement. The Copeland (1951) rule is an anonymous, neutral, variable population rule which satisfies overwhelming majority and does not admit minority overrides. But it does *not* satisfy reinforcement.

Anonymity. Let S_1 and S_2 be two neutral, $[0, 1]$-valued score systems. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two populations of voters. Define the variable-population voting rule F as follows: voters from $\mathcal{N}_1$ send signals (translated into real-valued scores) using the system S_1, whereas voters in $\mathcal{N}_2$ use S_2. All the scores from both populations are added together, and the alternative with the highest score is chosen. The rule F is neutral, satisfies reinforcement and overwhelming majority, and does not admit minority overrides. But F is *not* anonymous, because voters in $\mathcal{N}_1$ and $\mathcal{N}_2$ are treated differently.

References

Alcantud, J.C.R., Laruelle, A., 2013. To approve or not to approve: this is not the only question. Preprint.

Alós-Ferrer, C., 2006. A simple characterization of approval voting. Soc. Choice Welf. 27 (3), 621–625.

Arrow, K.J., Sen, A.K., Suzumura, K. (Eds.), 2002. Handbook of Social Choice and Welfare, Vol. I. North-Holland, Amsterdam.

Blackorby, C., Bossert, W., Donaldson, D., 2002. Utilitarianism and the theory of justice. Arrow et al. (2002), pp. 543–596.

Copeland, A., 1951. A 'reasonable' social welfare function. In: Seminar on Mathematics in Social Sciences. University of Michigan.

Dhillon, A., 1998. Extended Pareto rules and relative utilitarianism. Soc. Choice Welf. 15 (4), 521–542.

Dhillon, A., Mertens, J.F., 1999. Relative utilitarianism. Econometrica 67 (3), 471–498.

Dummit, D.S., Foote, R.M., 2004. Abstract Algebra, third ed. John Wiley & Sons Inc., Hoboken, NJ.

Fishburn, P.C., 1978. Axioms for approval voting: direct proof. J. Econom. Theory 19 (1), 180–185.

Fuchs, L., 2011. Partially Ordered Algebraic Systems. Dover, New York.

Gaertner, W., Xu, Y., 2012. A general scoring rule. Math. Social Sci. 63 (3), 193–196.

Macé, A., 2013. An axiomatization of range voting. Preprint.

Morkelyunas, A., 1981. On the choice from collections of sets. Mat. Metody v Sotsial. Nauk.—Trudy Sem. Protsessy Optimal. Upravleniya (14), 33–49. 93.

Myerson, R.B., 1995. Axiomatic derivation of scoring rules without the ordering assumption. Soc. Choice Welf. 12 (1), 59–74.

Nitzan, S., Rubinstein, A., 1981. A further characterization of Borda ranking method. Public Choice 36, 153–158.

Núñez, M., Laslier, J.F., 2013. Preference intensity representation: strategic overstating in large elections. Soc. Choice Welf. (in preparation).

Pivato, M., 2013. Variable-population voting rules. J. Math. Econom. 49 (3), 210–221.

Smith, J.H., 1973. Aggregation of preferences with variable electorate. Econometrica 41, 1027–1041.

Smith, W.D., 2000. Range voting, http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.32.9150.

Young, H.P., 1974a. An axiomatization of Borda's rule. J. Econom. Theory 9 (1), 43–52.

Young, H.P., 1974b. A note on preference aggregation. Econometrica 42, 1129–1131.

Young, H.P., 1975. Social choice scoring functions. SIAM J. Appl. Math. 28, 824–838.